Lecture Notes: Models of Industry Dynamics

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Outline

- **Lecture 1:** Three industry dynamic models with heterogenous firms
  - Jovanovic (1982): Selection and the evolution of industry
  - Hopenhayn (1992): Entry and exit in long-run competitive equilibrium
  - Ericson and Pakes (1995): Markov-perfect industry dynamic
- **Lecture 2:** Applications and computation of MPE
  - Quality ladder model (Pakes and McGuire (1994)),
  - Learning-by-doing (Benkard (2004), Besanko et al. (2010)),
  - Capacity dynamics (Besanko and Doraszelski (2004))

Jovanovic (1982): Selection and the evolution of industry

- **Stylized facts:**
  - The FSD is skewed to the right (e.g. log-normal).
  - Growth is negatively related to size (i.e. deviations for proportional theory of growth).
  - Failure rate is much higher for small firms.
  - Cohort size dispersion is increasing over time (i.e. more inequality).
- Intuition of the model:
  - Firms are uncertain about their productivity, and learn about it by staying in the industry ("passive" bayesian learning model).
  - At entry all firms look alike (common prior).
  - Overtime only "good" firms survive, and their size converges to their "true" efficient size (learning effect).
  - Bad firms decline in size, and exit the market early (selection effect).

Model

- Perfect competition between atomistic firms: p follows a known path (assume constant).
- Profits at time t conditional on information set $I(t)$:

$$\pi_t(I(t))) \;=\; \max_q pq - c(q)E[x_t|I(t)]$$

  where $c(q)$ is convex (i.e. DRS) and the efficiency level x_t is give by:

$$\begin{aligned} x_t \quad &= \quad g(\eta_t) = g(\theta + \epsilon_t), \\ \text{where} \quad &\theta \sim N(\bar{\theta}, \sigma_\theta^2) \\ &\epsilon_t \sim N(0, \sigma^2) \end{aligned}$$

- Learning and production decision:
  - At t_0 (i.e. entry date), new firms have the same common prior $\bar{\theta}$ about their productivity level and decide to produce $q(\bar{\theta})$.
  - At the end the each period ϵ_t is realized (i.e. x_t is observed), and firms update their beliefs about their type:

$$\bar{\eta}_{t+1} = \frac{1}{t - t_0} \sum_\tau \eta_\tau$$

– At the beginning of period $t+1$, active firms choose their production level based on $\bar{\eta}_{t+1}$:

$$q(\bar{\eta}_{t+1}) = \arg\max_{q} \pi(I(t+1))$$

- Dynamic decision problem:
  - The state at time t is summarized by $I(t) = \{\bar{\eta}_t, t - t_0\}$, plus the market price (assumed constant here).
  - Exit takes place at the end of the period (i.e. before production decision is made).
  - The present value of being active in the industry is given by the following Bellman equation:

$$V(I(t)) \;=\; \pi(I(t)) + \beta \int \max[W, V(I(t+1))] dF(\bar{\eta}_{t+1} | \bar{\eta}_t, t+1-t_0)$$

    Where W is the (constant) outside option.

- **Theorem 1:** (i) V is bounded and continuous, and (ii) V is strictly decreasing in $\bar{\eta}_t$.
- Therefore, firms' strategy is characterized by a threshold decision rule:

$$\text{Stay in if } \bar{\eta}_t \leq \gamma(t - t_0)$$

where $\gamma(t - t_0)$ is increasing in $(t - t_0)$, and converges to a constant $\bar{\gamma}$ as the age goes to infinity (i.e. the learning process is over).

- Implication for the probability of exit:
  - $\Pr(\text{Stay}|t - t_0) = \Pr(\bar{\eta}_t \leq \gamma(t - t_0))$
  - Since the threshold is increasing with age, the probability of staying in the market is increasing over time.
  - Firms are are more likely to exit when they are young.
  - This is due to learning: Young firms share common prior about their own productivity, and are more sensitive to changes in $\bar{\eta}_t$.

- **Equilibrium Entry/Exit (Theorem 3):** If $p_t = D_t(Q)$ is nondecreasing in t, and $q(p_t/x)$ is strictly concave in x, then the equilibrium price sequence is constant $p_t = \bar{p}$ for each t, and entry and exit occurs in equilibrium at each t.

  Key elements of the proof:

  - $x_t^* = E(x_{t+1}|I(t))$ is a random-walk.
  - When the supply function is concave in x, Jensen inequality holds:

    $$Eq(\bar{p}/x_{t+1}^*) < q(\bar{p}/x_t^*)$$

    In other words, successful firms grow at a declining rate (eventually zero growth).
  - Firms exit with positive probability: $\Pr(\bar{\eta}_t > \gamma(t - t_0))$.
  - Combining the two results: aggregate output from incumbents is decreasing. This violates the constant price.
  - To maintain $p_t = \bar{p}$ new firms must enter every period.

- **Implications of the model:**
  - Exit rate decreases with age.
  - Surviving firms are larger than exiting firms.
  - Successful firms growth faster:
    - If a firm receives an unexpectedly good news (e.g. low marginal cost shock) it revises its productivity up.
    - In the next period, this firm will have a higher expected productivity and will produce more.
    - Alternative explanation would rely on credit constraints: high profitable firms can use internal capital to finance growth.
  - The sequences of output tend to diverge:
    - Entrants are smaller and homogeneous.
    - The sequences of x_t^* are independent and converge to $\xi(\theta)$
    - The FSD for a given cohort tend to be more dispersed over time: the Gini coefficient increases over time (not necessarily monotonically).
    - The industry becomes more concentrated over time.
    - There is a positive relationship between concentration and profits (for large firm): The survival of very efficient firms create rents at the "top".

– The growth rate of young firms is larger and more variable than the one of older firms:
  * Learning process: x_t^* converges to a constant, so without aggregate shocks the growth rate of surviving firms converges to zero.
  * Selection bias: In the data, only survival firms are observed. Therefore, if we don't correct for this endogeneous selection, we will over-estimate the growth rate of young firms.

Entry/Exit and Firm Dynamics in Long Run Equilibrium, by Hugo Hopenhayn (1992)

- **Motivation:** Construct a dynamic equilibrium model of an industry that takes into account the following empirical facts:
  - In most US industries, the distribution of firm size is stationary (i.e. approximately equal entry/exit rates).
  - There exists substantial variation across industries in the entry/exit rates.
  - Most industries exhibit high turnover.
  - Firm-level uncertainty dominates aggregate uncertainty: entry/exit rates are not very cyclical.
- Key elements of the model:
  - Firm-level randomness: firms' productivity follow a first-order markov process.
  - No aggregate shocks: prices and aggregate outcomes are perfectly predictable.
  - Firms face a sunk cost c_e to enter the market, and incur a fixed cost c_f to stay-in.
  - Entry and exit occur in equilibrium.
  - Focus on a stationary equilibrium.

- **Model notation:**
  - Homogenous input and output markets:

$$p_t = D(Q_t)$$
$$w_t = W(N_t)$$

    where Q_t and N_t are aggregate output and input (i.e. labor) demand.
  - A firm of productivity ψ chooses output q to maximize each period profit taking as given prices:

$$\pi(\psi, p, w) = \max_q pq - c(q, \psi, w) - c_f \tag{1}$$

    where $c(\psi, w)$ is strictly decreasing in ψ, and c_f is a fixed cost.
  - Productivity evolves independently across firms according to the following Markov transition probability matrix:

$$\Pr(\psi_{t+1} = \psi' | \psi_t = \psi) = \Gamma(\psi', \psi) \tag{2}$$

    Note that for simplicity I assume that the productivity is following a discrete Markov chain (important: recursive process).
  - New entrants enter the market by paying cost c_e. After they enter their initial productivity is drawn from a distribution $\gamma(\psi)$.

- **State of the industry:**
  - At any point in time there is a continuum of firms in the industry with different productivity level. We define the structure of the industry by the distribution of firms across productivity levels: $\mu_t(\psi)$ is the number of firms of type ψ active at time t.
  - Importantly this vector is sufficient to describe all aggregate outcomes of the industry:

$$\text{Equilibrium price: } p_t = D\left(\sum_{\psi} \mu_t(\psi) q(\psi, p_t, w_t)\right)$$

$$\text{Equilibrium wage: } w_t = W\left(\sum_{\psi} \mu_t(\psi) n(\psi, p_t, w_t)\right)$$

  - We therefore define the state of the industry by μ_t.
  - An industry is stationary if $\mu_t = \mu_{t+1}$. Note: In a stationary industry aggregate prices are **constant**.

- **Entry/exit decisions:**
  - The problem of an incumbent firm is described by the following bellman equation:

$$v(\psi, z) = \pi(\psi, z) + \beta \max \left\{ 0, \sum_{\psi'} v(\psi', z)\Gamma(\psi'|\psi) \right\} \tag{3}$$

    where $z = \{p, w\}$. The solution to this problem is a simple threshold function $x(z)$:

$$x(z) = \inf \left\{ x \in \Psi \middle| \sum_{\psi'} v(\psi', z)\Gamma(\psi'|x) \geq 0 \right\}$$

    where Ψ is the set of possible productivity levels. In words the exit strategy of a firm is to stay active when the productivity is higher than $x(z)$, and exit otherwise.
  - For entrants the problem is decide wether the expected value of being in the industry given z is greater than the cost of entry:

$$c_e \leq v^e(z) = \sum_{\psi} v(\psi, z)\gamma(\psi) \tag{4}$$

    Let M the number of entrants in equilibrium.
  - Given the entry/exit strategies, we can define a law of motion for the industry state:

$$\mu_{t+1}(\psi') = \sum_{\psi > x(z)} \Gamma(\psi'|\psi)\mu_t(\psi) + M_t\gamma(\psi') \tag{5}$$

- **Industry equilibrium:**
  - It is useful to define the aggregate prices z as function only of μ (i.e. impose static competitive equilibrium). The value functions can then be re-written as function of ψ and μ only: $v(\psi, \mu)$ and $v^e(\psi, \mu)$.
  - **Definition of a stationary equilibrium:** An equilibrium is a distribution μ^*, a threshold x^*, a number of entrants M^*, aggregate output Q^* and input N^*, and prices p^*, w^* such that:
    1. Prices p^* and w^* clear the markets:

$$p^* = D(Q^*), \quad w^* = W(N^*)$$

       where $Q^* = \sum_\psi q(\psi, p^*, w^*)\mu^*(\psi)$, and $N^* = \sum_\psi n(\psi, p^*, w^*)\mu^*(\psi)$.
    2. Exit threshold is optimal:

$$x^* = \inf \left\{ x \in \Psi \middle| \sum_{\psi'} v(\psi', \mu^*)\Gamma(\psi', x) \geq 0 \right\}$$

    3. Free entry: M^* is such that,

$$v^e(\mu^*) \equiv \sum_\psi v(\psi, \mu^*)\gamma(\psi) = c_e.$$

4. Stationary industry structure:

$$\mu^*(\psi) = \sum_{\psi' > x^*} \Gamma(\psi, \psi')\mu^*(\psi') + M^*\gamma(\psi')$$

or alternatively in matrix form:

$$\mu^* = m(x^*, M^*) = M^* \left(I - \Gamma^*\right)^{-1} \gamma$$

where Γ^* is the equilibrium transition function, taking into account selection (i.e. exit).

– An alternative way of defining a stationary equilibrium is to find the values of (x^*, μ^*, M^*) such that:

$$x^* = \inf \left\{ x \in \Psi \middle| \sum_{\psi'} v(\psi', \mu^*)\Gamma(\psi', x) \leq 0 \right\} \tag{6}$$

$$c_e = \sum_{\psi} v(\psi, \mu^*)\gamma(\psi) \tag{7}$$

$$\mu^* = m(M^*, x^*) \tag{8}$$

– Existence proof steps:

* Stationary distribution $m(M^*, x^*)$ is well defined (i.e. $\downarrow$ in x and $\uparrow$ in M).
* Let $M_1(x)$ be the mass of entrants defined from exit rule:

$$\sum_{\psi'} v(\psi', m(x, M_1(x))\Gamma(\psi', x) = 0$$

* Let $M_2(x)$ be the mass of entrants defined from zero profit condition:

$$c_e = \sum_{\psi} v(\psi, m(x, M_2(x))\gamma(\psi)$$

* An equilibrium is defined as an x^* such that $M_1(x^*) = M_2(x^*)$.

* $M_1(x)$ and $M_2(x)$ are continuous nondecreasing functions.
* At $x = 1$, $M_1(x) > M_2(x)$: If all firms exit the market, the predicted number of entrants must be higher under M_1 and M_2.
* If entry cost is high enough, it is possible that $M_1(x) > M_2(x)$ for all x (i.e. indeterminacy).
* For low enough c_e, excess entry occurs at $x = 0$ and $M_2(0) > M_1(0)$.

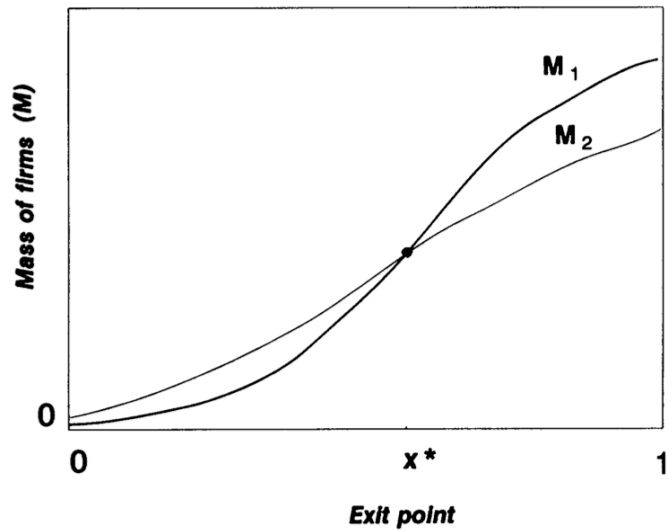


FIGURE 1.—Existence of stationary equilibrium.

– The paper suggests a procedure to solve for an equilibrium. For all x find the values of $M_1(x)$ and $M_2(x)$ such that:

$$x = \inf \left\{ x \in \Psi \middle| \sum_{\psi'} v(\psi', m(M_1, x))\Gamma(\psi', x) \geq 0 \right\} \tag{9}$$

$$c_e = \sum_{\psi} v(\psi, m(M_2, x))\gamma(\psi) \tag{10}$$

An equilibrium is a x^* such that $M_1(x^*) = M_2(x^*)$.

- **Equilibrium properties of the model**
  - Under general conditions the distribution of productivity among survival firms of cohort t, denoted by $\tilde{\mu}_t$, is increasing in t.
    - Key assumption: Productivity is positively correlated over time, $\partial\Gamma(\psi'|\psi)/\partial\psi > 0$.
    - This also implies that average value, size and survival probability will be the **lowest** for most recent entrants.
    - In other words, the firm size distribution (or equivalently productivity) is stochastically increasing in the age of the cohort.
    - **Note:** Age does not have any direct effect. Size is a sufficient statistic for ψ (i.e. larger firms are longer lived and will remain larger before exit). To introduce age effects we need to augment the model with an experience effect (e.g. $\Gamma(\psi'|\psi, t)$).
  - Since the firm size distribution is one-to-one with the productivity distribution, this implies that the model predicts that the firm size distribution is increasing in firms' age (similar to Dunne-Roberts-Samuelson).

– More comparative statics:
  * x^* and M^* are both decreasing in c_e: High entry cost industries have low turnover rate.
    · Graphically, $\uparrow c_e$ shifts $M_2(x)$ downwards, and equilibrium x^* is lower.
  * An increase in c_e might have positive or negative effect on firms value: (i) raise profits (i.e. price effect), (ii) reduce x^* so that less productive firms stay in (selection effect).
  * The effect of c_f is ambiguous.
    · $\uparrow c_f$ shifts both $M_1(x)$ and $M_2(x)$ downward.
    · If profits are separable in ψ, larger fixed costs lead to larger firms.
    · The opposite can be true if the profit function is not separable in ψ (see example in the paper).

Dynamic Investment Game with Entry/Exit (Ericson and Pakes (1995))

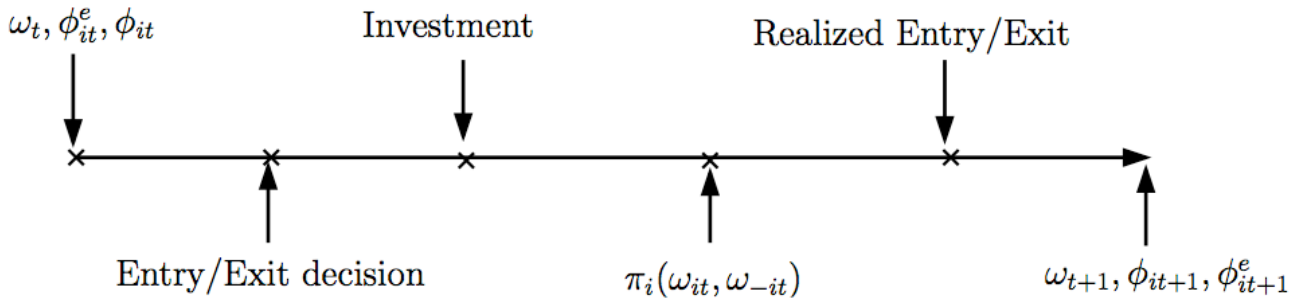
- **Introduction:**
  - Central element: Productivity is stochastically controlled by firms' investments.
  - Investment is strategic: Markov Perfect Equilibrium (MPE).
  - Model is general enough that it can be applied in various contexts:
    1. Quality ladder model (Pakes and McGuire (1994)),
    2. Learning-by-doing (Benkard (2004), Besanko et al. (2010)),
    3. Capacity dynamics (Besanko and Doraszelski (2004)),
    4. Horizontal mergers (Gowrisankaran (1999)),
    5. Advertisement (Doraszelski and Markovich (2005)).

    $*$ See Doraszelski and Pakes (2006) for a review.

Model

- **Actions:**
  1. Static actions: Price/quantity game. Lead to a (unique) static payoff function $\pi_i(\omega_i, \omega_{-i})$.
  2. Dynamic actions: Entry/Exit, Investment $x_i(\omega_i, \omega_{-i})$.
- **Assumptions:** $\pi_i(\omega_i, \omega_{-i})$ is continuous, bounded, and increasing in ω_i.
- **State space:**
  - Firm i's productivity: $\omega_i \in \Omega = \{1, 2, ..., \bar{\omega}\}$ (public information).
    * ω_i can represents a quality index, a marginal cost shifter, capacity, etc.
  - Entry cost: $\phi_i^e \sim F^e(.)$ (private information).
  - Scrap value: $\phi_i \sim F(.)$ (private information).
  - Industry state: $\omega \in S = \{\omega_1, ..., \omega_n | \omega_i \in \Omega, n \leq \bar{n}\}$
  - **Note:** $|S|$ grows exponentially in $\bar{n}$ (curse of dimensionality).

- **Assumptions:** Payoffs and strategies are **symmetric** and **anonymous**.
  - Symmetry: $f_i(\omega_i, \omega_{-i}) = f_j(\omega_i, \omega_{-i})$.
  - Anonymous: $f(\omega_i, \omega_{-i}) = f(\omega_i, \omega_{\mu(-i)})$.
  - Where $\mu(-i)$ is any reordering of indexes $-i$.
- The symmetry and anonymity assumptions reduce dramatically the industry state space. Two ways of writing the industry state:
  1. $\omega \in S^0 = \{\omega_1, ..., \omega_n | \omega_i \in \Omega, \omega_1 > \omega_2 > ... > \omega_n, n \leq \bar{n}\} \subset S$
  2. $\omega \in S^0 = \{s_1, ..., s_{\bar{\omega}} | \omega_i \in \Omega, \sum_\omega s_\omega \leq \bar{n}\} \subset S$
- **Timing:**



- **Investment:**
  - Firms invest x_i to raise the probability of getting a productivity gain ν next period:

$$\Pr(\omega_i' = \omega_i + \nu_i | \omega_i, x_i) = P_\nu(\nu_i | x, \omega_i)$$

    where $\nu_i \in \{0, 1\}$ and $p_\nu(\nu_i = 1 | x_i = 0, \omega_i) = 0$.
  - Assumption: $p_\nu(\nu = 1 | x, \omega)$ is strictly increasing in x_i.
  - Example (Pakes and McGuire (1994)):

$$p_\nu(\nu = 1 | x, \omega) = p_\nu(\nu = 1 | x) = \frac{\alpha x}{1 + \alpha x}$$

- **Exit:**
  - At the beginning of the period incumbents observe scrap value of equipments ϕ_i and decide to exit at the end of the period (i.e. all incumbents receive $\pi(\omega_i, \omega_{-i})$).
  - Since ϕ_i is private information, we can characterize the exit strategy either by an exit cutoff $\bar{\phi}(\omega_i, \omega_{-i})$, or by the continuation probability $r(\omega_i, \omega_{-i}) = \Pr(\phi < \bar{\phi}(\omega_i, \omega_{-i}))$.

- **Entry:**
  - Each period there are $\mathcal{E}$ potential entrants who live **only** one period.
  - Potential entrants privately observe an entry cost ϕ^e and decides to enter or not at the end of the period (one period set-up cost).
  - Entrants choose to invest x^e in order to improve their productivity next period.
  - Entry strategy: Two representations: Optimal cutoff point $\bar{\phi}^e(\omega_i, \omega_{-i})$, or entry probability $r^e = \Pr(\phi < \bar{\phi}^e(\omega_i, \omega_{-i}))$.
- **State-to-state transitions:**
  - Individual state transition:

$$\omega_i' = \omega_i + \underbrace{\nu_i}_{\text{productivity gain}} - \underbrace{\eta}_{\text{aggregate shock}},$$

    where $\eta \in \{0, 1\}$ and $\Pr(\eta = 1) = \delta$.
    Note: The depreciation shock η can either be industry-wide or firm specific.
  - Industry state transition probability matrix: $q(\omega_{-i}'|\omega, \eta)$ is the perceived transition probabilities of competitors' states (i.e. beliefs).

- **Value functions:**
  - Incumbent's problem:

$$V(\omega_i, \omega_{-i}, \phi) \;=\; \pi(\omega_i, \omega_{-i}) + \max\Big[-x(\omega_i, \omega_{-i}) + \beta E\big(V(\omega_i', \omega_{-i}', \phi')|\omega_i, \omega_{-i}, x_i\big), \phi\Big]$$

Or, prior to observe ϕ:

$$E_\phi(V(\omega_i, \omega_{-i}, \phi)) \equiv v(\omega_i, \omega_{-i}) \;=\; \pi(\omega_i, \omega_{-i}) + (1 - r_i)E[\phi|\beta EW(\omega) - x_i < 0] + r_i\Big[-x_i + \beta \sum_\nu W_i(\nu|\omega) p_\nu(\nu|x_i)\Big]$$

Where:

* $W_i(\nu|\omega) = \sum_{\omega_{-i}', \eta} v(\omega_i + \nu - \eta, \omega_{-i}') q(\omega_{-i}'|\omega_i, \omega_{-i}, \eta) p_\eta(\eta)$,
* $\omega = (\omega_i, \omega_{-i})$.
* r_i is the optimal continuation probability:

$$r_i(\omega, \omega_{-i}) = F\big(\beta \sum_\nu W_i(\nu|\omega) p_\nu(\nu|x_i) - x_i\big)$$

* x_i solves the following KT-FOC:

$$x_i\Big(\beta \sum_\nu W(\nu|\omega) \frac{\partial p_\nu(\nu|x_i)}{\partial x_i} - 1\Big) = 0$$

– Entrant's problem:

$$V^e(\omega, \phi^e) = \max_{\chi^e, x^e} \chi^e(-\phi^e - x_i^e + \beta \sum_{\nu} W_e(\nu|\omega) p_\nu(\nu|x^e)$$

Where W_e is the continuation value of an entrant (i.e. starting at ω_e), x^e is defined as before. The entry probability is given by:

$$r^e(\omega_i, \omega_{-i}) = F^e\Big(- x^e(\omega) + \beta \sum_{\nu} W(\nu|\omega) p_\nu(\nu|x^e(\omega)\Big)$$

- **Equilibrium:** A MPE equilibrium is a set of functions $\{x, r, x^e, r^e\}$ such that:

1. The policy functions solve the incumbents and entrants problems given beliefs $q(.|.)$.
2. The perceived aggregate transition probabilities $q(\omega'_{-i}|\omega, \eta)$ are consistent with the optimal response of all agents.

Properties of the equilibrium

- Existence of a pure strategy equilibrium:
  - Continuity of the scrap value and entry cost distributions ensure the existence of unique entry/exit strategies in probability space (i.e. perfect-bayesian equilibrium, see Doraszelski and Satterthwaite (2005) for the exact conditions of profits and F).
  - As long the $p_\nu(.|x)$ satisfies some regularity conditions such that the continuation value is concave, there exits a unique best-reply investment function (see Ericson and Pakes (1995)).
- Uniqueness: Very difficult to show, apart for some special cases.

- Characterization:
  - There exists a pair $(\bar{n}, \bar{\omega})$ such that with probability one:
    - there will never be more than $\bar{n}$ firms active,
    - we will never observe an active firm with $\omega_i > \bar{\omega}$ (i.e. $x_i(\bar{\omega}, \omega_{-i}) = 0, \forall \omega_{-i}$).

    Therefore, the state space is always **finite** and we can compute an equilibrium.
- The equilibrium defines a finite state Markov chain with transition probability matrix $Q(\omega', \omega)$. Therefore there exists at least one recurrent class of states $R \in S^0$, such that:
  - with probability one each sample path $\{\omega_t\}$ will enter R,
  - and once in R the process will stay within it forever.

  Note: Since R is typically much smaller and S^0, if we can identify it, we can reduce considerably the computation burden (see Pakes and McGuire (2001)).

Computing MPE Nash equilibrium—
Gauss-Jacobi VS Gauss-Seidel algorithms

- **Example:**
  - Static entry game with 2 players and incomplete information.
  - Payoff of entering:

$$\pi_i = \alpha_i - \theta E(N_{-i}) + \epsilon_i = \alpha_i - \theta \tilde{p}_{-i} + \epsilon_i$$

  - If ϵ_i is iid extreme value, the best-response function is given by:

$$p_i = R(\tilde{p}_{-i}) = \frac{\exp(\alpha_i - \theta \tilde{p}_{-i})}{1 + \exp(\alpha_i - \theta \tilde{p}_{-i})}$$

    and a Nash equilibrium is such that $p_i = R(p_{-i})$ for all is.

- **Gauss-Jacobi**:

it 0: Guess p_i^0.

it k: Let $\tilde{p}_j = p_j^{k-1}$, and update strategies using the best-response mapping:

$$p_i^k = R(\tilde{p}_{-i})$$

test: Stop if $||p^k - p^{k-1}|| < \delta$, else update $k + 1$.

- **Gauss-Seidel**:

it 0: Guess p_i^0.

it k.1: Let $\tilde{p}_{-1} = p_{-1}^{k-1}$, and update strategies using the best-response mapping for player 1:

$$p_1^k = R(\tilde{p}_{-1})$$

...

it $k.j$: Let $\tilde{p}_i = p_i^{k-1}$ for $i > j$, and $\tilde{p}_i = p_i^k$ for $i < j$. Update strategies using the best-response mapping for player i:

$$p_j^k = R(\tilde{p}_{-j})$$

...

test: Stop if $||p^k - p^{k-1}|| < \delta$, else update $k + 1$.

- The Guass-Seidel usually converges faster, but both algorithm can converge to different equilibria (or not converge at all...).

- Alternatives:
  - Damping: $p_i^k = \alpha R(\tilde{p}_{-i}) + (1 - \alpha)p_i^{k-1}$
  - Random updating orders (in the Gauss-Seidel algorithm).

- **Pake-McGuire Algorithm (modified) - Gauss-Jacobi:**
  - Initialization: Set $\underline{\omega}$ and $\bar{\omega}$. How?

    Solve the monopoly problem for arbitrary small $\underline{\omega}$ and large $\bar{\omega}$, and set:

$$\bar{\omega} = \arg\max_{\omega} x^m(\omega) = 0$$
$$\underline{\omega} = \arg\min_{\omega} r^e(\omega) \simeq 0$$

    Also, we need to set a value for $\bar{n}$ "large enough" (see below).
  - Objects in memory at each iteration:

$$M^l(\omega_i, \omega_{-i}) = \{v^l(\omega_i, \omega_{-i}), x^l(\omega_i, \omega_{-i}), x^{e,l}(\omega_i, \omega_{-i}), r^l(\omega_i, \omega_{-i}), r^{e,l}(\omega_i, \omega_{-i})\}$$

– Iteration l: For each $\omega_i \in \Omega$

1. **Compute** $q(\omega'_{-i}|\omega_i, \omega_{-i}, \eta)$. Recall:

$$q(\omega'_{-i}|\omega_i, \omega_{-i}, \eta) = \prod_{j \neq i} p^{l-1}(\omega'_j|\omega_i, \omega_{-i}, \omega) \prod_{k=1}^{E'} p_\nu(\nu_k|x^{e,l-1}) r^{e,l-1}(E', \nu)$$

Where,

$$p^{l-1}(\omega'_j|\omega_i, \omega_{-i}, \eta) = \begin{cases} (1 - r_j^{l-1}) & \text{if } \omega'_j = \emptyset \\ r^{l-1} p_\nu(1|x_j^{l-1}) & \text{if } \omega'_j = \omega_j + 1 - \eta \\ r^{l-1} p_\nu(0|x_j^{l-1}) & \text{if } \omega'_j = \omega_j - \eta \end{cases} \quad (11)$$

$$r^{e,l-1}(E', \omega_i, \omega_{-i}) = \binom{E}{E'} r^{e,l-1}(\omega_i, \omega_{-i})^{E'} (1 - r^{e,l-1}(\omega_i, \omega_{-i}))^{E-E'} \quad (12)$$

In practice, we need to identify the set of "reachable" states $S(\omega_{-i}) \subset S$. How?

* For incumbents, $\omega'_j \in \{\emptyset, \omega_j + 1 - \eta, \omega_j - \eta\}$
* For entrants, $\omega'_j \in \{\emptyset, \omega^e + 1 - \eta, \omega^e - \eta\}$

If there are n active incumbents and E potential entrants, the dimensionality of $S(\omega_{-i})$ is 3^{n-1+E}.

This process is then repeated for each $\omega_{-i} \in \Omega^{\bar{n}-1}$. **Computing** $q(\omega'_{-i}|\omega_i, \omega_{-i}, \eta)$ **is the main source of the curse of dimensionality problem.**

2. Compute expected continuation values:

$$W^l(\nu|\omega_i,\omega_{-i}) = \sum_{\omega'_{-i},\eta'} v^{l-1}(\omega_i+\nu_i-\eta')q(\omega'_{-i}|\omega_i,\omega_{-i},\eta')p_\eta(\eta') \tag{13}$$

$$W^{e,l}(\nu|\omega) = \sum_{\omega',\eta'} v^{l-1}(\omega^e+\nu_i-\eta')q(\omega'|\omega,\eta')p_\eta(\eta') \tag{14}$$

3. Update policy functions (i.e. best-responses):

   (a) Investment decisions:

$$x^l(\omega_i,\omega_{-i})\Big(\beta(W^l(1)-W^l(0))\frac{\partial p_\nu(1|x^l)}{\partial x^l}-1\Big)=0$$

$$x^{e,l}(\omega_i,\omega_{-i})\Big(\beta(W^{e,l}(1)-W^{e,l}(0))\frac{\partial p_\nu(1|x^{e,l})}{\partial x^{e,l}}-1\Big)=0$$

   (b) Entry/Continuation probabilities:

$$r^l(\omega_i,\omega_{-i}) = F\big(\beta\sum_\nu W^l\nu|\omega_i,\omega_{-i})p_\nu(\nu|x_i)-x^l\big)$$

$$r^{e,l}(\omega) = F^e\big(\beta\sum_\nu W^{e,l}(\nu|\omega)p_\nu(\nu|x_i)-x^{el}\big)$$

4. Update value function:

$$v^l(\omega_i, \omega_{-i}) = \pi(\omega_i, \omega_{-i}) + (1 - r^l)E[\phi|\beta W^l(\omega_i, \omega_{-i}) - x^l < 0] + r^l\Big[- x^l + \beta \sum_{\nu} W^l(\nu|\omega_i, \omega_{-i}) p_\nu(\nu|x^l)\Big]$$

– Stop if $||v^l(\omega_i, \omega_{-i}) - v^{l-1}(\omega_i, \omega_{-i})|| < \delta$.

- Endogenous $\bar{n}$?
  - Solve the model with $\bar{n}^k$
  - If $\min_\omega r^e(\omega|n(\omega) = \bar{n}) > \epsilon$, recompute the equilibrium with $\bar{n}^{k+1} = \bar{n}^k + 1$.

- **Computation burden:**
  - Number of states $\#S$: Raise the cost of the outer-loop over states (ω_i, ω_{-i}).
  - Transition probabilities $q(\omega'_{-i}|\omega_i, \omega_{-i}, \eta)$ (inside loop to find transient states)
    - $\#S$ grows exponentially in $\bar{n}$.
    - $\#S^0$ grows at polynomial rate in $\bar{n}$.
    - $\#S$ and $\#S^0$ grow exponentially in $\bar{\omega}$: Binding in models with multi-product or multi-state firms (e.g. differentiated products or horizontal mergers).
    - Even when the transient states $S(\omega_{-i})$ can be "pre-computed", the transition probabilities must be re-computed for each new iteration of the policy functions.
- **Alleviating the computational burden:**
  - Stochastic algorithm: Pakes and McGuire (2001).
  - Continuous-time representation: Doraszelski and Judd (2004).
  - Parallel representation: Benkard (2004)
  - Other methods: Approximation methods (e.g. Keane and Wolpin (1997))

Besanko and Doraszelski (2004) Capacity dynamics and endogenous asymmetries in firm size

- **Stylized facts**
  - Market structure is persistent
  - Most industries exhibit important asymmetries: Dominant firm(s) competing with fringes.
  - How heterogeneous initial conditions emerge in the first place?
  - Cournot vs Bertrand: Equivalent market-structure?

- **Model set-up**
  - Duopoly
  - Homogenous products: $D(P) = a - bP$.
  - Heterogeneous (discrete) capacities:

$$0 \leq q_i \leq \bar{q}_i$$

    and industry structure: $\omega_t = \{\bar{q}_{1t}, \bar{q}_{2t}\}$.
  - Two models of competition:
    1. Quantity: Cournot
    2. Price competition: Edgeworth-Bertrand (see for instance Deneckere & Kovenock)
  - Investment: Investing x affects the probability of capacity improvement

$$\Pr(\Delta\bar{q}_{1t}|\bar{q}_{1t}, x_{1t}) = \begin{cases} \frac{(1-\delta)\alpha x_{1t}}{1+\alpha x_{1t}} & \text{If } \Delta q_{1t} = 1, \\ \frac{(1-\delta)}{1+\alpha x_{1t}} + \frac{\delta\alpha x_{1t}}{1+\alpha x_{1t}} & \text{If } \Delta q_{1t} = 0, \\ \frac{\delta}{1+\alpha x_{1t}} & \text{If } \Delta q_{1t} = -1 \end{cases}$$

    This is in the interior of the capacity grid. Boundary states: $\bar{q} = M$ and $\bar{q} = 0$.

- **Investment decision:**
  - Bellman equation:

$$V_1^{\tilde{x}}(\omega) = \pi_1(\omega) + \max_{x_1 \geq 0} \left\{ -x_1 + \beta \sum_{\vec{q}_1'} W_1^{\tilde{x}}(\vec{q}_1'|\omega) \Pr(\vec{q}_1'|\bar{q}_1, x_1) \right\}$$

    where $W_1^{\tilde{x}}(\bar{q}_1) = \sum_{\vec{q}_2'} V_1^{\tilde{x}}(\bar{q}_1, \bar{q}_2) \Pr(\vec{q}_2'|\bar{q}_2, \tilde{x}_2)$.
  - Kuhn-Tucker condition:

$$-1 + \beta \sum_{\vec{q}_1'} W_1^{\tilde{x}}(\vec{q}_1'|\omega) \frac{\partial \Pr(\vec{q}_1'|\bar{q}_1, x_1)}{\partial x_1} \geq 0$$

  - Closed-form best-response function:

$$x_1(\omega) = R_1(\tilde{x}|\omega) = \max \left\{ 0, \frac{-1 + \sqrt{\beta\alpha \left((1-\delta) \left(W_1^{\tilde{x}}(\bar{q}_1 + 1) - W_1^{\tilde{x}}(\bar{q}_1) \right) + \delta \left(W_1^{\tilde{x}}(\bar{q}_1) - W_1^{\tilde{x}}(\bar{q}_1 - 1) \right) \right)}}{\alpha} \right\}$$

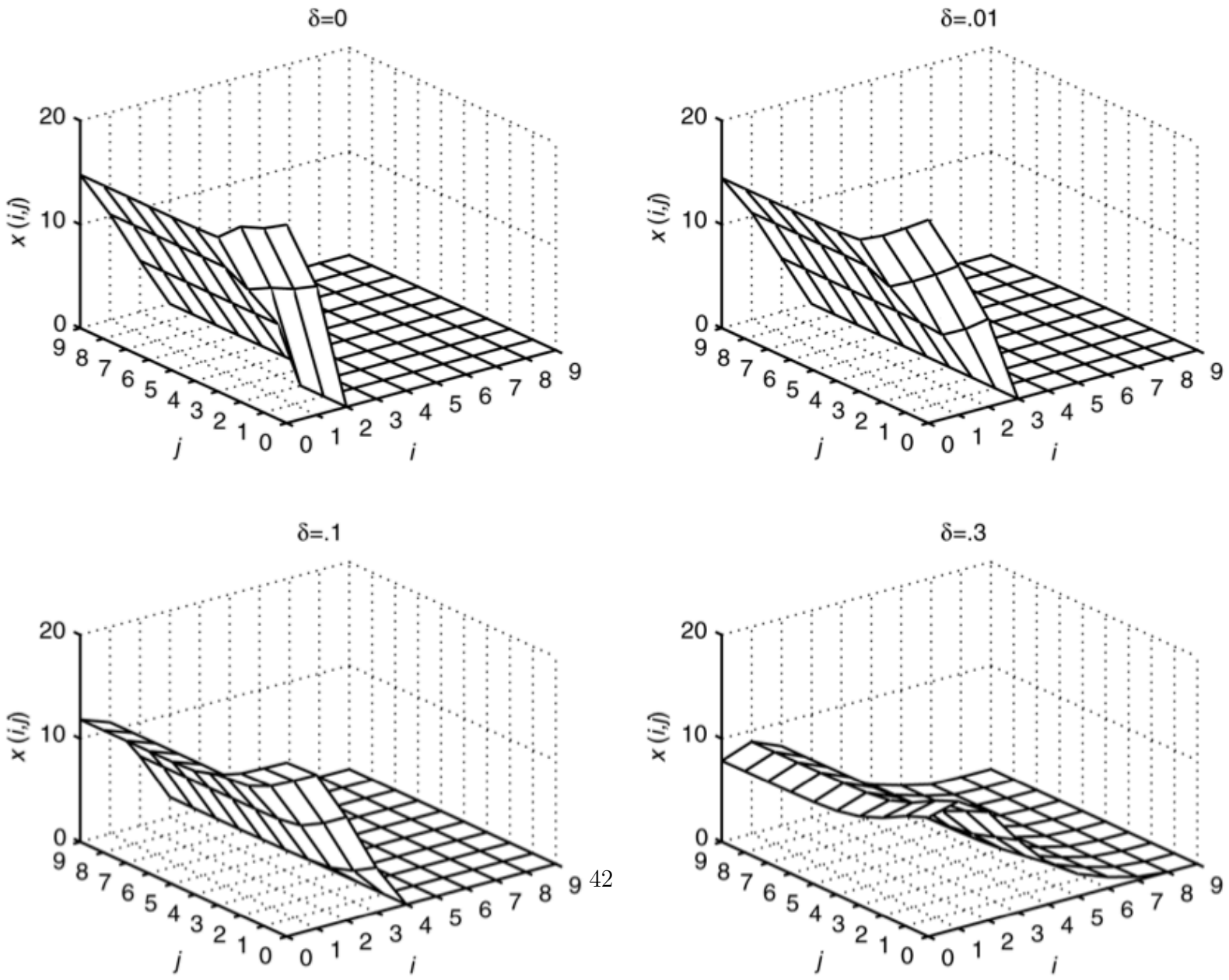
- **MPE Nash equilibrium:**
  - $x_i(\omega) = R_1(x_j(\omega)|\omega)$ for all $i, j = \{1, 2\}$.
  - $V_i^x(\omega) = \pi_i(\omega) + -x_i + \beta \sum_{\vec{q}_1'} W_i^x(\vec{q}_1'|\omega) \Pr(\vec{q}_i'|\bar{q}_i, x_i)$ for all $i, j = \{1, 2\}$.

- **Algorithm:**
  - Guess: $\tilde{x}(\omega)$ and $\tilde{V}(\omega$ for all ω
  - Iteration k:
    - Update $x_i(\omega)$ using $\tilde{x}$ and $\tilde{V}$
    - Update $V_i(\omega)$ using x and $\tilde{V}$
  - Stopping rule: $x_i(\omega) = \tilde{x}_i(\omega)$ and $V_i(\omega) = \tilde{V}_i(\omega)$.
- **Simulations:**
  - Vary the depreciation rate δ, holding fixed the other parameters. Why?
  - δ measures the degree of investment irreversibility. Study the role of "credibility" on long-run industry distribution.

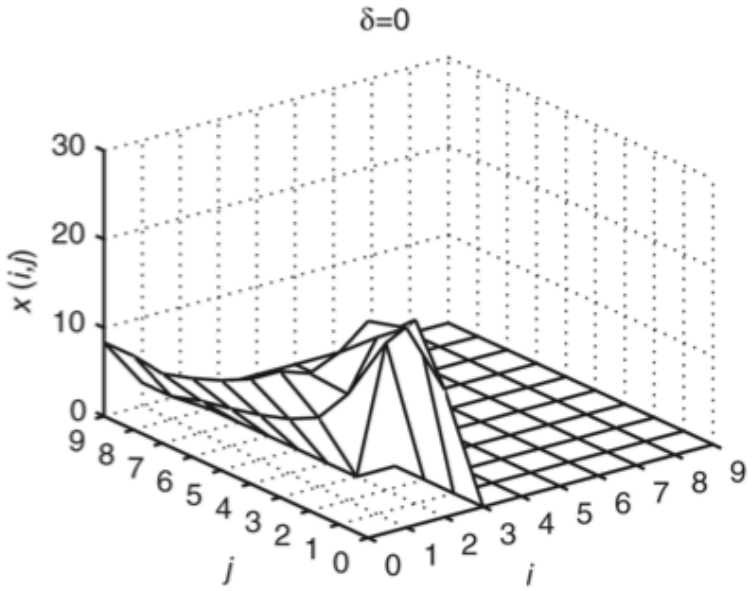
- **Simulation results**

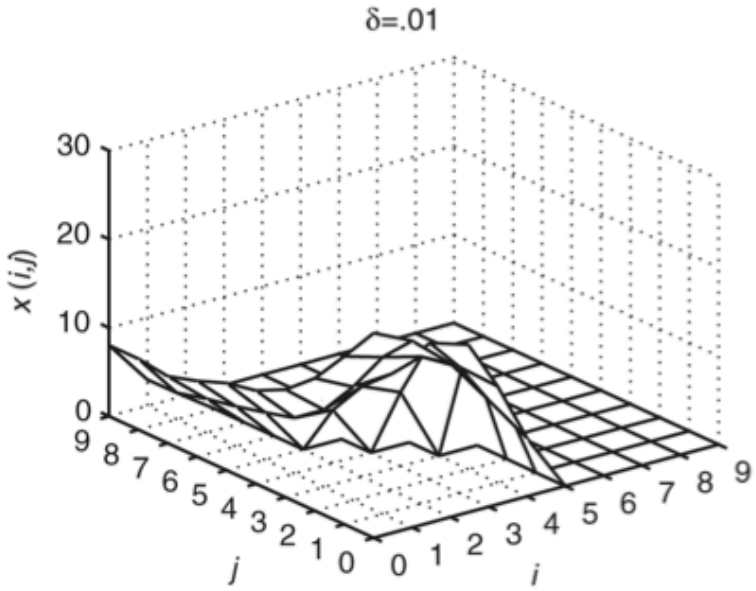
  1. Quantity game
     - If $\delta = 0$: Firms invest at constant rate until they reach Cournot: $x_1 = 0$ if $\bar{q}_1 > 3$.
     - If $\delta > 0$: Firms over-invest (again +/- constant) to maintain the Cournot quantity in the long-run. As a result, firms hold idle capacity, the average long-run capacity levels are increasing in δ, and the industry structure tend to be symmetric.
  2. Pricing game
     - Unlike in the Cournot model, investment functions depend crucially on competitors' capacity levels: Firms "give-up" quickly when opponent's level is larger than 3.
     - When firms are symmetric, their investment levels increase: Firm engage in a "pre-emption" race to establish a higher capacity level.
     - Investments are increasing in the depreciation fractor.
     - Why? With price competition, small capacity firms are easily shut-down, and large capacity firms earn monopoly power on the residual demand segment.

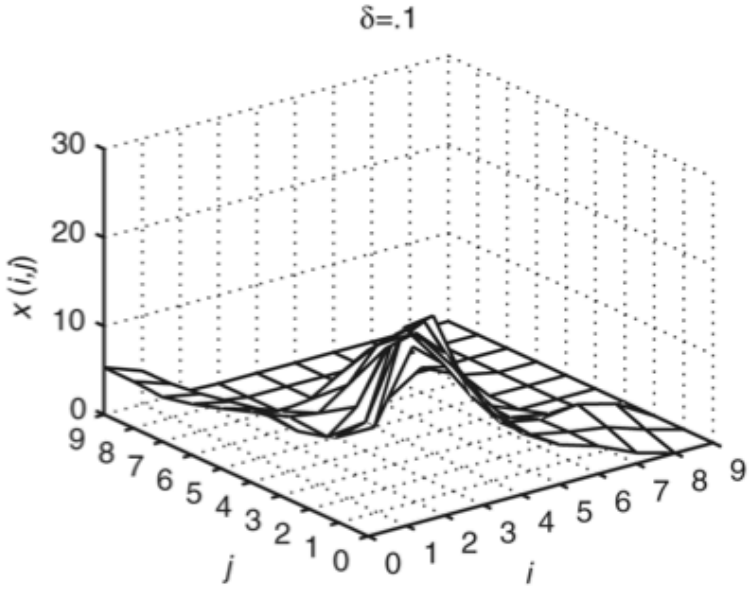
POLICY FUNCTION $x(i, j)$: QUANTITY COMPETITION

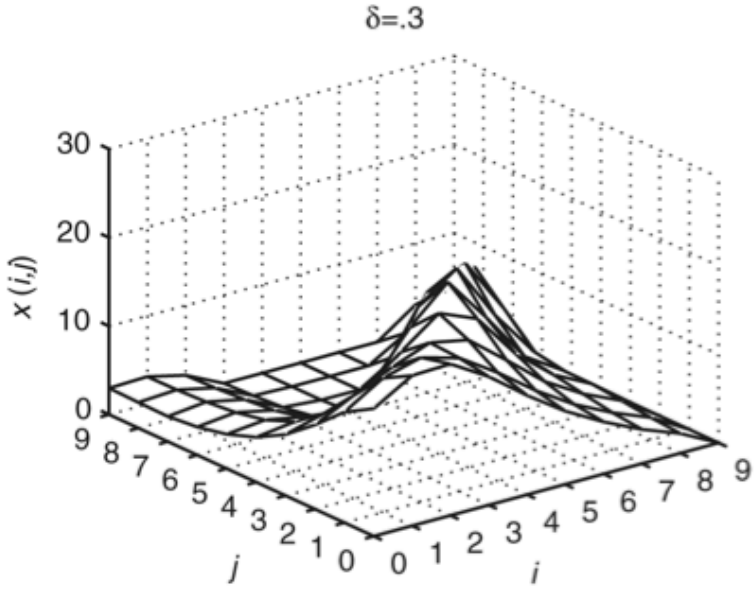


POLICY FUNCTION $x(i, j)$: PRICE COMPETITION

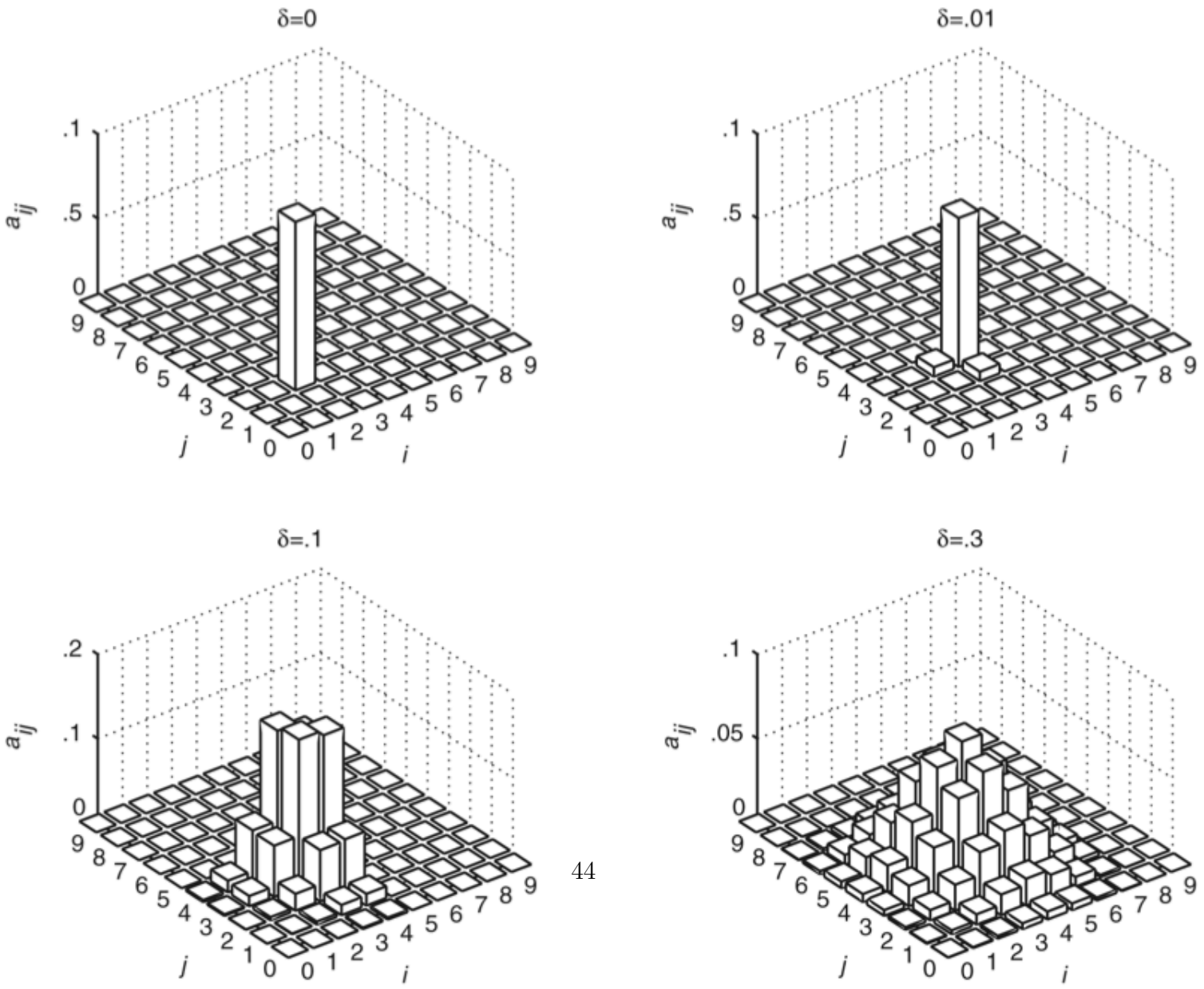




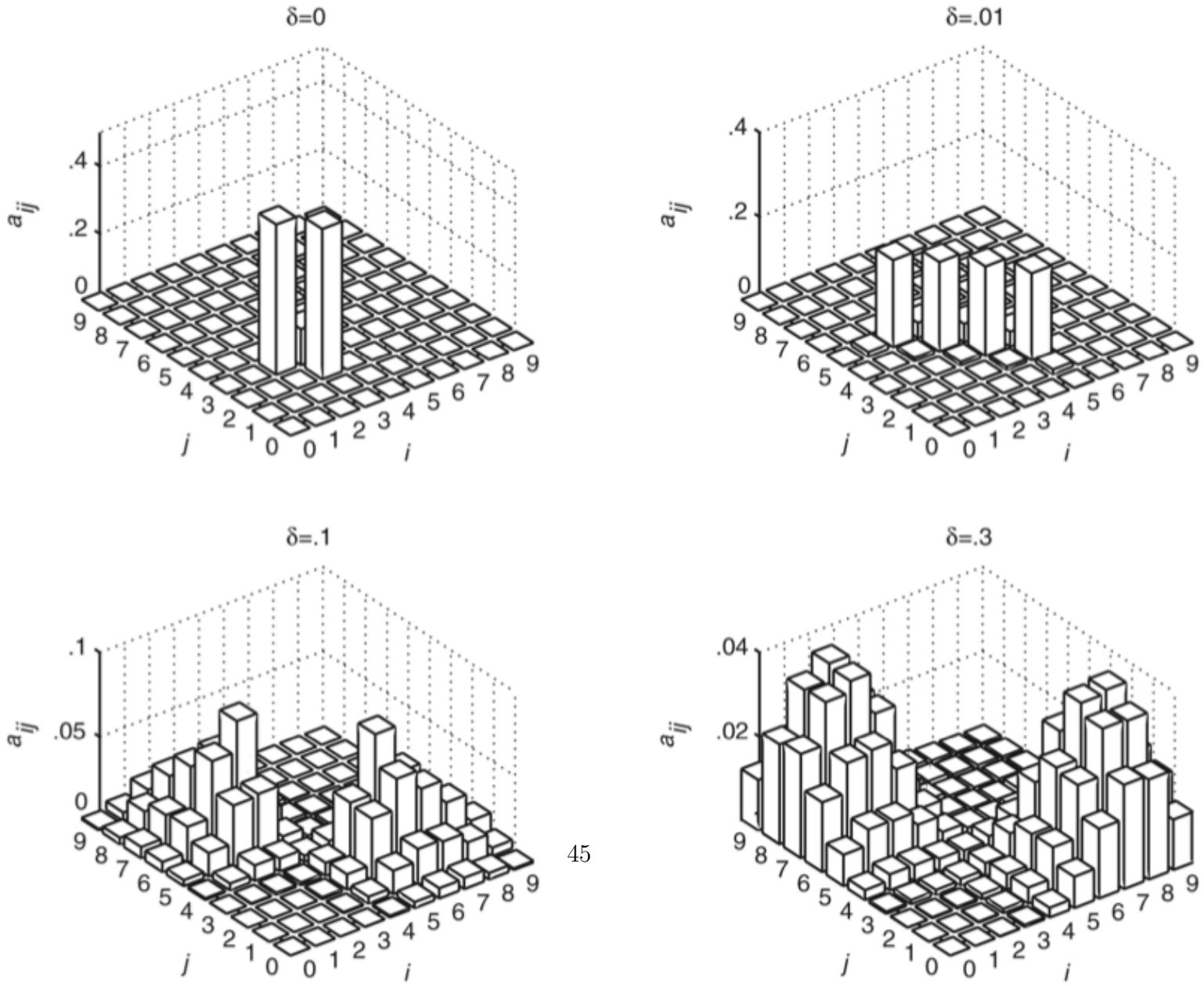




TRANSIENT DISTRIBUTION AFTER $T = 25$ PERIODS WITH INITIAL STATE $i_0 = j_0 = 1$: QUANTITY COMPETITION



TRANSIENT DISTRIBUTION AFTER $T = 25$ PERIODS WITH INITIAL STATE $i_0 = j_0 = 1$: PRICE COMPETITION



Benkard (2000) and Benkard (2004): Learning and forgetting in aircraft production

- **Motivation:**
  - Large empirical literature on measuring learning-by-doing in production
  - Traditional models focus on linear knowledge accumulation function:

$$L_t = A_t E_t^{\theta}$$

    where experience (E_t) accumulates over time:

$$E_t = E_{t-1} + 1, \text{ if } q_t > 0$$

  - Previous studies have documented limited presence of learning in the context of military contracts (i.e. $\approx$ production over-time).
  - Commercial production is **much more** volatile. Depreciation of knowledge stock is an important issue (e.g. a sudden drop if orders can have decrease productivity).
  - **Definition:** In a learning/forgetting knowledge capital accumulation, experience accumulates stochastically:

$$E_t = \delta E_{t-1} + q_t.$$

– **Case study:** The Lockheed L-1011

* 250 units produced between 1970-1984
* Four basic models:
  - Model 100-300 are similar and use comparable technologies - 80% of production.
  - Model 500 was introduced in 1978, and exhibit $\neq$ charact. - 20% of production.
* Three parameters of interest:
  - Learning rate (θ)
  - Depreciation (δ)
  - Knowledge spillover (i.e. introduction of model 500).
    Complete or incomplete pass-through?

– Drop in labor productivity following production slowdown (150th unit)
  * Decline in demand (oil shock)
  * Introduction of Model 500 (incomplete spillover)
  * Strike of 1977
– Evolution of labor requirement and annual output:

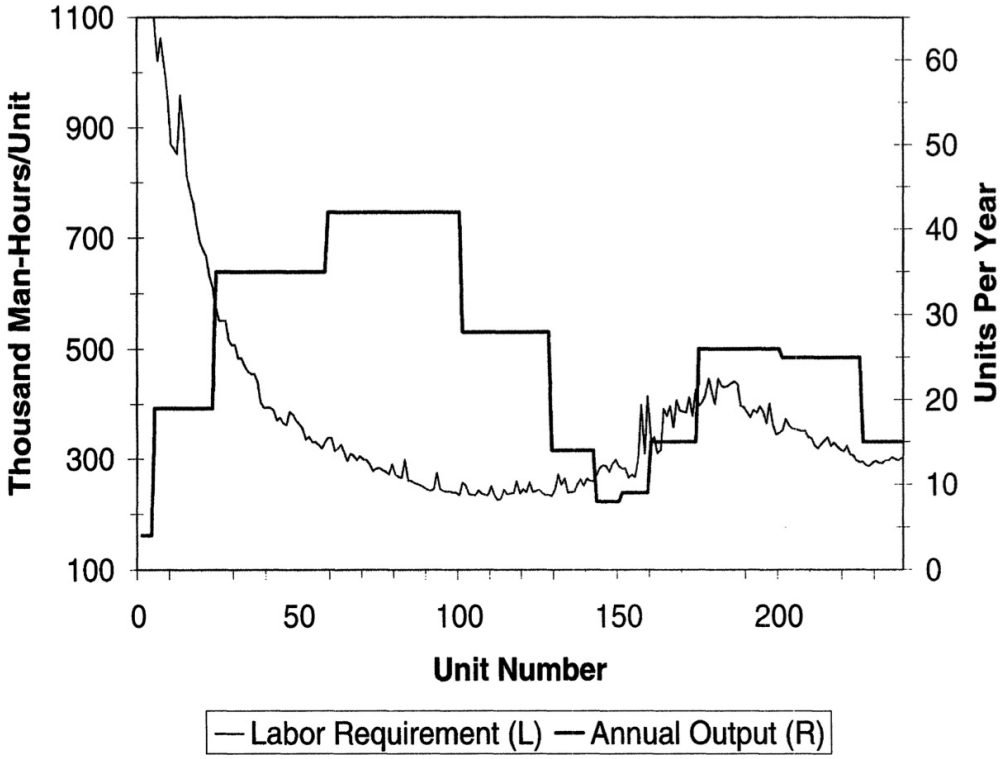


FIGURE 1. L-1011 PRODUCTION: DIRECT LABOR REQUIREMENT AND YEARLY OUTPUT

- **Production function and learning:**
  - Leontief production function in material and labor:

$$q = \min \left\{ G(L, E, \bar{K}, S, \epsilon), H(M, E, \bar{K}, S, \nu) \right\}$$

  - Data on production and labor units is available at the **unit** plane level.
  - Assume that the labor-requirement function to produce one unit (i.e. inverse of G) is Cobb-Douglas:

$$\log L_i = \log A(\bar{K}) + \theta \log E_i + \gamma_0 \log S_i + \log \epsilon_i$$

    where $A(\bar{K})$ is constant over time, and S_i is the line speed (must be inferred).
  - Experience accumulates for each product $j = \{100 - 300, 500\}$:

$$E_{j,i} = \delta E_{j,i-1} + q_{j,i-1} + \lambda q_{-j,i-1}$$

    where $q_{j,i-1}$ is the number of units produced up to unit i.
  - If $\lambda = 1$, learning accumulation is symmetric, and total output/experience matters to determine the productivity of a unit of labor.

- **Timing assumptions:**
  - Line-speed is set at the beginning of the period to meet demand
  - ϵ_i and ν_i are observed
  - Firm set material and labor optimally according to the labor requirement equation.
- **Identification:**
  - If ϵ_i is serially correlated, E_i and S_i are correlated with ϵ_i. Similarly, with imperfect spillover, both experience levels are included.
  - Instrumental variables:
    - Demand shifters: Lagged oil prices, GDP, time-trend.
    - Cost shifters: Lagged aluminum prices.
  - General model is estimated by GMM.

TABLE 1—TRADITIONAL LEARNING MODEL REGRESSIONS

	ln A	θ	γ_0	γ_1	Time	Adj.+	Adj.−	SSR	ρ_e	L.R.
Basic regressions										
1. Units 1–112	7.90	−0.51	—	—	—	—	—	1.36	0.73	30%
	(0.06)	(0.01)							(0.04)	
2. Units 1–238	7.16	−0.29	—	—	—	—	—	15.0	0.97	18%
	(0.08)	(0.02)							(0.02)	
Line speed										
3.	6.51	−0.35	0.95	−0.20	—	—	—	11.0	0.92	21%
	(0.21)	(0.02)	(0.17)	0.03					(0.03)	
Calendar time										
4.	6.03	−1.08	−0.04	0.004	1.16	—	—	5.4	0.56	53%
	(0.15)	(0.04)	(0.13)	(0.025)	(0.07)				(0.04)	
Adjustment cost										
5.	6.75	−0.31	0.67	−0.16	—	0.07	−0.04	15.8	0.43	19%
	(0.26)	(0.03)	(0.25)	(0.05)		(0.03)	(0.02)		(0.06)	
N = 238	TSS = 33.7									

Notes: All regressions are 2SLS. Instruments (Z_j) are present and lagged demand shifters (various world GDP measures, the price of oil, and a time trend; see text) and present and lagged cost shifters (U.S. wage rate, aluminum price). L.R. is the implied learning rate.

TABLE 2—TRADITIONAL LEARNING MODEL REGRESSIONS: INPUT PRICES AND DISECONOMIES OF SCOPE

	ln A	θ	γ_0	γ_1	Wage	P_{AL}	P_{oil}	Scope	SSR	ρ_e	L.R.
Diseconomies of scope											
6.	7.35	−0.49	0.49	−0.10	—	—	—	0.55	2.4	0.70	29%
	(0.10)	(0.01)	(0.08)	(0.02)				(0.02)		(0.04)	
Oil price											
7.	5.88	−0.54	1.36	−0.27	—	—	0.27	—	9.3	0.83	32%
	(0.21)	(0.03)	(0.17)	(0.03)			(0.04)			(0.04)	
Input prices											
8.	−15.9	−0.52	0.45	−0.09	8.68	0.50	—	—	10.0	0.81	30%
	(3.37)	(0.03)	(0.18)	(0.03)	(1.32)	(0.09)				(0.04)	
N = 238	TSS = 33.7										

Results from model with learning and forgetting

- The learning rate, -0.65, is higher in the full specification.
- High depreciation rate: On a yearly basis, 60% of beginning of the year knowledge capital survives.
- Incomplete spillover: Only 70% of experience accumulated on other lines of product is transmitted into productivity gains (i.e. production of 500 versus $100 - 300$).
- Rejection of the economies of scope model.
- Implication: Producing two models reduce overall productivity in the long-run.

  "If we assume values of $\lambda = 0.70$ *and* $\theta = 0.63$*, then the first* 500 *produced required approximately 25 percent more labor than model* 1 *alone requires. If Lockheed chose to produce both* $-500s$ *and* $-1s$ *in approximately equal numbers for the rest of the program, then labor requirements would continuously run approximately 11 percent higher than for the single-model case."*

	ln A	θ	γ_0	δ	λ	Wage	P_{AL}	P_{oil}	Scope	SSR	GMM(p)	ρ_ε	L.R.
Scope													
11. [$S_N^* = 7.7$]	7.91	−0.71	0.13	0.954	0.73	—	—	—	−0.03	3.1	0.62	0.45	39%
	(0.01)	(0.02)	(0.10)	(0.004)	(0.17)				(1.92)			(0.05)	
Oil price													
12. [$S_N^* = 9.0$]	7.72	−0.72	0.17	0.946	0.79	—	—	−0.003	—	3.3	0.89	0.51	39%
	(0.01)	(0.02)	(0.05)	(0.002)	(0.05)			(1.13)				(0.05)	
Input prices													
13. [$S_N^* = 7.9$]	7.63	−0.70	0.14	0.952	0.77	0.06	0.009	—	—	2.9	0.78	0.46	38%
	(0.13)	(0.01)	(0.08)	(0.003)	(0.04)	(6.36)	(2.96)					(0.05)	
$N = 238$	$TSS = 33.7$												

Notes: All regressions in this table use the HAC-IV method described in the text. Instruments (Z_i) are present and lagged demand shifters (various world GDP measures, the price of oil, and a time trend; see text) and present and lagged cost shifters (U.S. wage rate, aluminum price). S_N^* is the optimal bandwidth used in estimating the GMM covariance and optimal weight matrices. L.R. is the implied learning rate.

Model fit

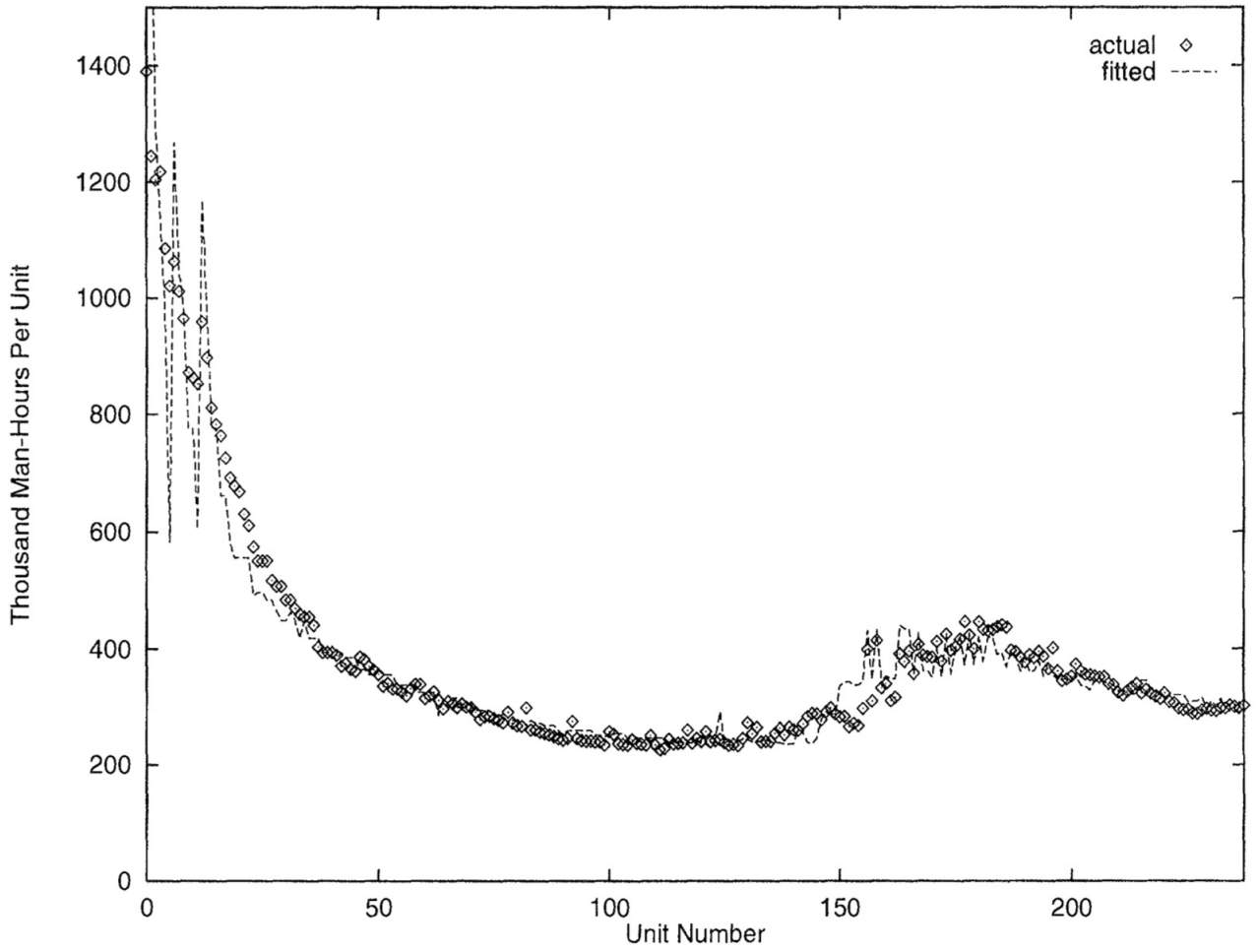
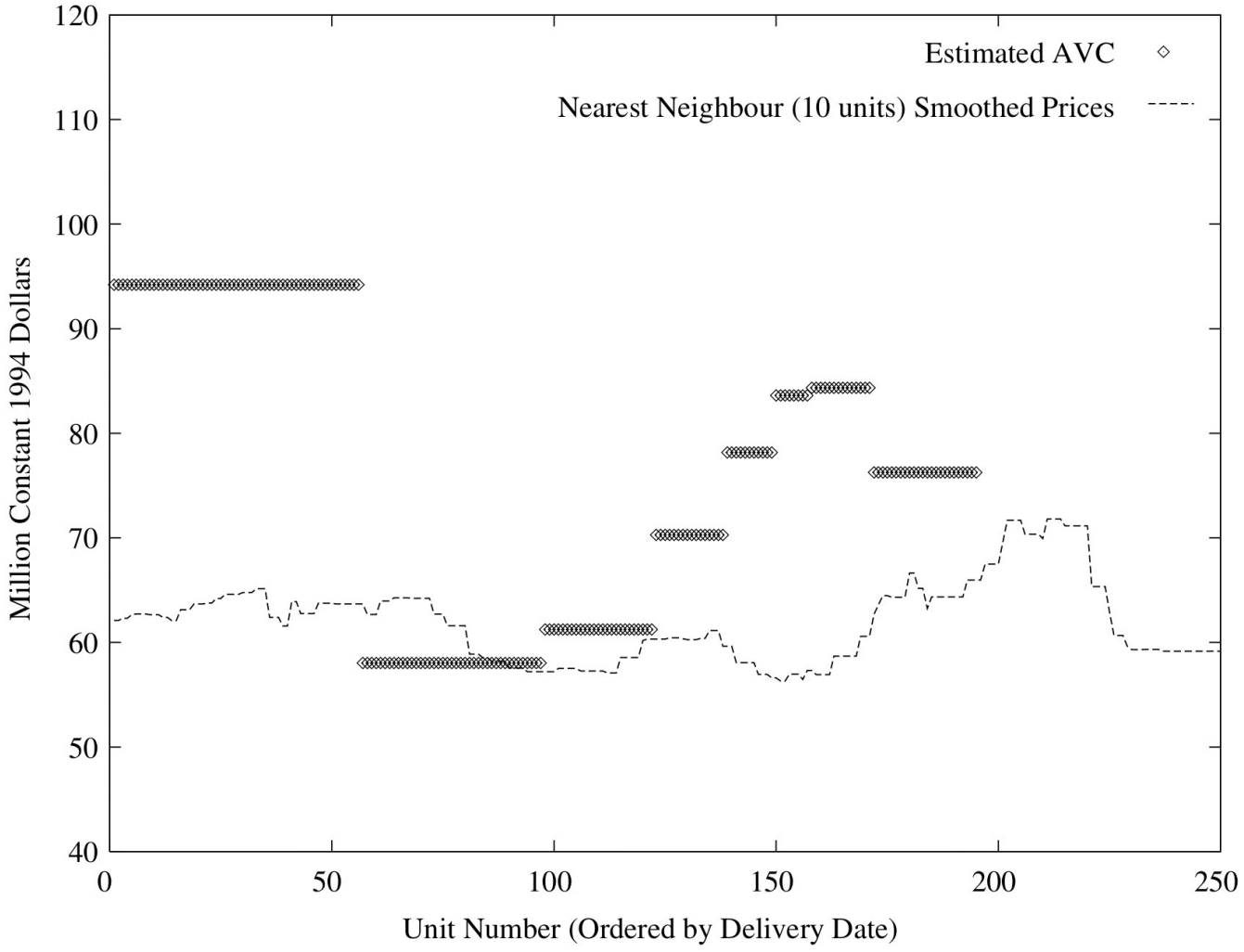


FIGURE 4. ORGANIZATIONAL FORGETTING AND INCOMPLETE SPILLOVERS (REGRESSION 10)

- **Motivation (numerical paper):**
  - Large theory literature: Impact of the "learning-by-doing" technologies on market structure and price competition.
  - Intuition: If costs are decreasing with experience, firms have an incentive to earn negative profits in the short-run (e.g. product introduction phase) in order to move "down" the learning curve.
  - Strategic incentives: Firms can strategically generate price wars to gain market shares in the future ($\sim$ accusation of predatory pricing behavior).
  - Stylized facts about aircraft production:
    * Prices are frequently smaller than average costs.
    * The industry is highly concentrated: Few new products/firms enter between 1969-1994.
    * Costs are more volatile than prices.

- **Prices and average cost:**



- **General Model:** Multi-product oligopolistic competition.
  - Product characteristics: $\omega_{jt} = \{E_{jt}, \mu_j, \xi_{jt}\} \in \Omega$.
    - Experience level: $E_{jt} \in \mathcal{E}$. Stochastically increasing in the production decision q_{jt}.
    - Product type: $\mu_j \in \{small, medium, large\}$. Fixed until exit.
    - Product quality: $\xi_{jt} \in \{\xi_1, ..., \bar{\xi}\}$. Evolves stochastically (and exogenously) according to a finite markov chain.
  - Product portfolio of firm i: $\mathcal{J}_i = \{\omega_{i1t}, ..., \omega_{iJ_it}\}$
  - Aggregate industry demand state: $M_t \in \{M_1, ..., \bar{M}\}$ (i.e. stochastic market size).
  - Industry state (s): Number of active firms in each state (i.e. Ω^J) + market size M.
- **Timing and actions:**
  1. Industry random states are realized: M, ξ_{ij}.
  2. Incumbents privately observe the scrap value of each product: $\Phi_{ij} \sim U[\underline{\Phi}, \bar{\Phi}]$.
  3. Product exit decisions are made: $\chi_{ij} = \{0, 1\}$.
  4. Incumbents and one (1) potential entrant observe privately a new product development cost x_j^e, and decide to launch a new product or not:
$$\chi_i^e = \mu_k, \quad \text{where } \mu_k \in \{0, \mu_s, \mu_m, \mu_l\}$$
  5. Production decisions are made for active products: q_{ij}.

- Bellman equation:

$$V(i,s,M) = \max_{\chi^e,\{\chi_j,q_j\}_{j\in\mathcal{J}_i}} \Big\{ -\sum_k 1(\chi^e = k)x_k^e + \sum_{j\in\mathcal{J}_i}\big[\chi_j\Phi_j + (1-\chi_j)\pi_j(i,s,q_j,M)\big] + \sum_{i',s',M'} V(i',s',M')P(i',s',M'|i,s,M,q,\chi,\chi^e)\Big\}$$

- Entrants problem is similar (except for production decision).
- Static profit: Cournot competition with differentiated products.

$$\pi_j(i,s,q,M) = p_j(i,s,q,M)q_j - c_j(i,q_j)$$

- **Computation limitations:** Impossible to solve if $J_i > 1$. Simplifications:
  - Restrict to single product firms: Ignore joint production decision.
  - Potential entrant product "type" is exogenous: Firms have the option to enter with product type μ_k with probability $p_\mu(k)$ (i.e. iid over time).
  - Restrict the number of firms/products to $\bar{n} = 4$.
  - ∗ The dimension the industry state is $13 = 3 \times \bar{n} + 1$, and $\#S \simeq 7M$.

- **Estimation/Calibration:**
  - **Cost function:** (Estimated using Lockead L-1011 accounting data)
    * Labor demand:

$$\log L_{jt} = \log A + \theta \log E_{jt} + \gamma \log S_{jt} + \epsilon_{jt}$$

      where $E_{jt} = \delta E_t + q_t$, and S_t is the scale of production.
    * Learning/Forgetting process: $\mathcal{E} = \{1, 10, 20, 40, 70, 110, 165\}$ and transitions are given by:

$$E_{t+1} = \begin{cases} |E^*_{t+1}|^+ & \text{with probability } \frac{E^*_{t+1} - |E^*_{t+1}|^-}{|E^*_{t+1}|^+ - |E^*_{t+1}|^-} \\ |E^*_{t+1}|^- & \text{with probability } 1 - \frac{E^*_{t+1} - |E^*_{t+1}|^-}{|E^*_{t+1}|^+ - |E^*_{t+1}|^-} \end{cases}$$

    * The scale and the learning parameters are held constant across time and products.
    * A is assumed to be exactly proportional to the plane size.
    * Total variable cost is then:

$$c_j(i, q_j) = c_0 q_j + c_1 w L_j(q_j, E_j)$$

∗ Parameter estimates:

Cost parameters

Parameter	Explanation	Value
A	Labour cost intercept	7·73 (0·01)
γ	Returns to scale	0·11 (0·17)
δ	Depreciation of experience	0·613 (0·023)
θ	Learning parameter	–0·63 (0·03)
	(Implied learning rate)	36%
W	Wage rate	$20/h
FC	Fixed costs	$200 million/year
TCF	Total variable cost/labour cost	6·0
TCC	Total variable cost intercept	36·2
	Cost/plane-size ratio	1·0
x_1^l, x_1^h	Type 1: entry cost distribution	$2·5–$3·5 billion
x_2^l, x_2^h	Type 2: entry cost distribution	$3·3–$4·6 billion
x_3^l, x_3^h	Type 3: entry cost distribution	$4·4–$6·2 billion

– **Demand function:**

  * Durability: Assume the existence of a complete rental market (avoid the problem of solving for a dynamic demand of demand).
  * Use new aircraft prices: Assume that rental prices are proportional to new prices.
  * Nested Logit specification: Assume a continuum of "consumers" making independent decisions (i.e. airline companies divisions).
    · Nest 1: All new wide-bodied aircrafts.
    · Nest 2: Used and other categories.
  * Indirect static utilities:

$$u_{ijt} = X_{ijt}\beta - \alpha p_{jt} + \xi_{jt} + \eta_{igt} + (1-\lambda)\epsilon_{ijt}$$

  * Berry's inverse representation:

$$\xi_{jt} = \ln s_{jt} - \ln s_{0t} - X_{jt}\beta + \alpha p_{jt} - \lambda \ln s_{jt|g}$$

– Transition matrix for demand state variables (ξ and M): Cell frequencies (i.e. 5 and 3 respectively).

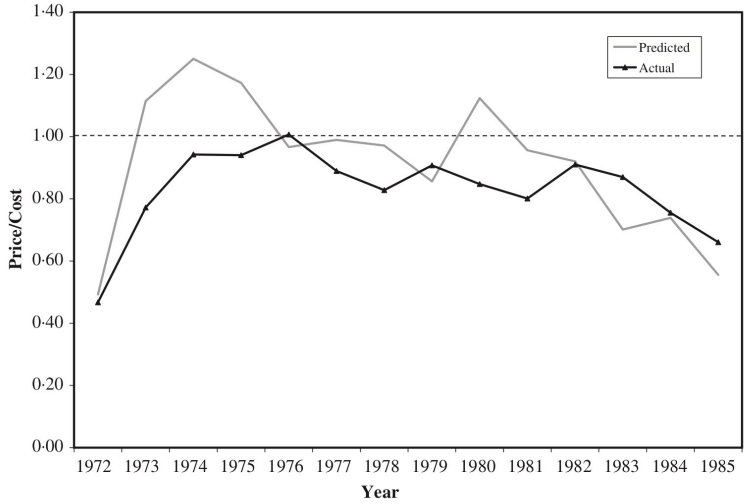
– Parameter estimates:

TABLE 3

Demand and other parameters

Parameter	Explanation	Value
λ	Group corr. parameter	0·77 (0·18)
α	Price coefficient	−0·024 (0·002)
μ	Discrete plane types (small, medium, large)	$\{-2{\cdot}6, -2{\cdot}2, -1{\cdot}6\}$
$P(\mu^e)$	Entry type distribution (small, medium, large)	$(0{\cdot}50 \quad 0{\cdot}38 \quad 0{\cdot}12)$
ξ	Discrete plane qualities	$\{-0{\cdot}90, -0{\cdot}40, 0{\cdot}11, 0{\cdot}61\}$
$\Delta\xi$	Transition matrix for quality	$\begin{pmatrix} 1{\cdot}00 & 0{\cdot}04 & 0{\cdot}033 & 0{\cdot}000 \\ 0{\cdot}00 & 0{\cdot}44 & 0{\cdot}233 & 0{\cdot}200 \\ 0{\cdot}00 & 0{\cdot}48 & 0{\cdot}667 & 0{\cdot}800 \\ 0{\cdot}00 & 0{\cdot}04 & 0{\cdot}067 & 0{\cdot}000 \end{pmatrix}$
M	Discrete market sizes	(10,339 10,929 11,519)
ΔM	Transition matrix for market size	$\begin{pmatrix} 0{\cdot}895 & 0{\cdot}143 & 0{\cdot}000 \\ 0{\cdot}105 & 0{\cdot}786 & 0{\cdot}200 \\ 0{\cdot}000 & 0{\cdot}071 & 0{\cdot}800 \end{pmatrix}$
β	Firm's discount factor	0·925
(Φ^l, Φ^h)	Range of scrap values	(300m$, 700m$)

- **Results:**
  - **Price-Cost margins:** Compare model predictions with Lockhead accounting data.
    - Predictions are matching the fact that $P/AC < 1$ for most of the life-cycle of the product.
    - Model provide a rational explanation: It is optimal for Lockhead to increase its production level in order to increase the likelihood of obtaining productivity gains (never realized).
    - Low P/AC ratios are observed when: (i) more products are offered, (ii) incumbents' products are higher quality, (iii) incumbents have more experience.



– **Industry dynamics simulations:**

* Simulations with initial condition=1969 match fairly well the observed market structures:

* Invariant distribution of market structure highly concentrated.

* The model predict that the long-run distribution of plane types will change in favor of smaller planes (why?).

* In the long-run 60% of products are profitable.

* The life-expectancy of products is highly variable (median = 22 years).

TABLE 5

Model simulations and historical industry characteristics 1969–1994

	Concentration ratios		
	Observed	I.C. simulation	Invariant distribution
1-Plane	0·44	0·47	0·40
S.D.	0·20	0·17	0·10
2-Plane	0·68	0·75	0·69
S.D.	0·14	0·13	0·11
1-Firm	0·55	—	—
S.D.	0·17		
2-Firm	0·82	—	—
S.D.	0·12		
Market size:			
# Planes	4·4	3·5	3·8
S.D.	1·2	0·8	0·4
# Firms	3·4	—	—
S.D.	0·7		
	Distribution of plane types		
	Observed	I.C. simulation	Invariant distribution
Small	0·56	0·55	0·75
Medium	0·23	0·21	0·23
Large	0·21	0·24	0·02

– **Policy Experiment:** Concentration restrictions reduce consumer surplus ($\simeq 1\%$). Why? Firms cannot reach their optimal experience level, which cause excessive entry.

Learning-by-doing and organizational forgetting (Besanko et al. (2010))

- **Starting point:**
  - Cabral & Riordan (1994): LbyD promotes market dominance
  - From there, it might seem like adding forgetting should have a pro-competitive effect
  - However, the paper shows that the opposite is more likely true: Incorporating forgetting induces larger asymmetries and more concentration.
    - "Forgetting" opens the possibility of weakening your opponent by cutting prices
    - This combination of an "offensive" and "defensive" strategy is more pronounced with organizational forgetting
  - Forgetting is a source of equilibrium multiplicity
    - Need to develop a method to trace the equilibrium correspondence.

- **Model:**
  - Duopoly without entry/exit
  - Discrete experience level: $e_n \in \{1, 2, ..., M\}$
  - Each period, one client comes to the market
  - Accumulation: $e'_n = e_n + q_n - f_n$. Where:
    - $q_n \in \{0, 1\}$ indicates sales/no-sales for firm n with probability $D_n(\mathbf{p})$.
    - $f_n \in \{0, 1\}$ indicates depreciation (i.e. forgetting).
  - Forgetting probability:
$$\Delta(e_n) = 1 - (1 - \delta)^{e_n}$$
  - Learning curve:
$$c(e_n) = \kappa f(e_n)^{\eta}$$
    where $f(e_n) = m$ if $e_n > m$, where m denotes bottom of the learning curve.
  - Demand/sales probability: $D_n(\mathbf{p}) = \frac{1}{1+\exp\left(\frac{p_n - p_{-n}}{\sigma}\right)}$

- Bellman equation:

$$V_n(\mathbf{e}) = \max_{p_n} D_n(p_n, p_{-n}(\mathbf{e}))(p_n - c(e_n)) + \beta \sum_{k=1}^{2} D_k(p_n, p_{-n}(\mathbf{e}))\bar{V}_{nk}(\mathbf{e})$$

  where $\bar{V}_{nk}(\mathbf{e}) = E\left(V_n(\mathbf{e}')|\mathbf{e}, q_k = 1\right)$.

- First-order condition:

$$D_n(\mathbf{e}) + \frac{\partial D_n}{\partial p_n}(p_n - c(e_n)) + \beta \sum_k \frac{\partial \bar{V}_{nk}(\mathbf{e})}{\partial p_n}\bar{V}_{nk}(\mathbf{e}) = 0$$

$$D_n(\mathbf{e})\left(\sigma - (p_n - c(e_n)) + V_n(\mathbf{e}) - \beta\bar{V}_{nn}(\mathbf{e})\right) = 0$$

- Symmetric MPE (arbitrary for firm 1):

$$F^1(e|p^*, V^*) = -V_1^*(\mathbf{e}) + D_1^*(\mathbf{e})(p_1^*(\mathbf{e}) - c(e_1)) + \beta \sum_k D_k^*(\mathbf{e})\bar{V}_{1k}^*(\mathbf{e}) = 0$$

$$F^2(e|p^*, V^*) = D_1^*(\mathbf{e})\left(\sigma - (p_1^*(\mathbf{e}) - c(e_n)) + V_1^*(\mathbf{e}) - \beta\bar{V}_{11}^*(\mathbf{e})\right) = 0$$

  for all states. In matrix form:

$$F(p^*, V^*) = \begin{pmatrix} F^1(p^*, V^*) \\ F^2(p^*, V^*) \end{pmatrix} = 0$$

  is $2M^2$ system of non-linear equations.

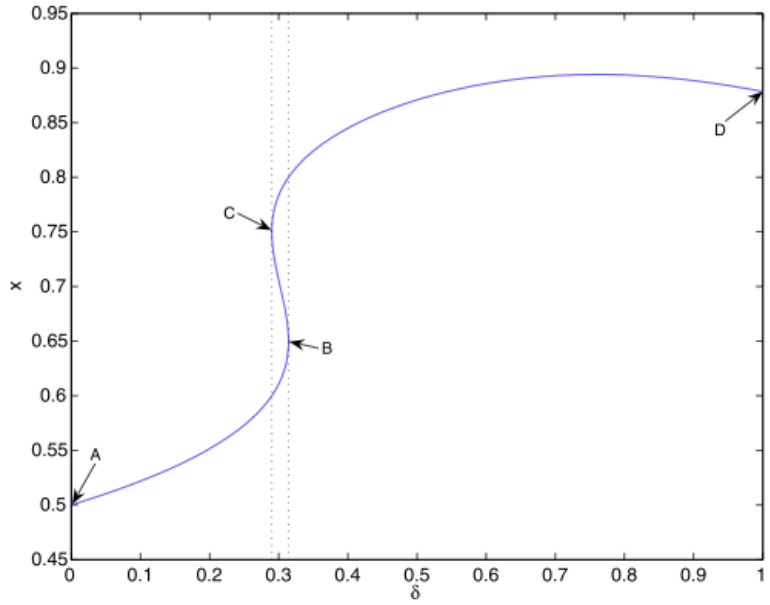
- **Computation: Homotopy method**
  - **Goal:** For each ρ, need to explore the equilibrium correspondence:

$$F^{-1}(\rho) = \{V^*, p^*, \delta | F(V^*, p^*; \delta, \rho) = 0, \delta \in (0,1)\}$$

  - Example:

$$F(x, \delta) = -15.289 - \frac{\delta}{1+\delta^4} + 67.50x - 96.923x^2 + 46.154x^3 = 0$$

How to trace the correspondance?



– The algorithm iterates on a differential equation:

  * Lets index each point of the correspondence by s: $(x(s), \delta(s))$
  * For any s the following total differential equation applies:

$$\frac{\partial F(x(s),\delta(s))}{\partial x}x'(s) + \frac{\partial F(x(s),\delta(s))}{\partial \delta}\delta'(s) = 0$$

  * This differential equation is solved by setting:

$$x'(s) = \frac{\partial F(x(s),\delta(s))}{\partial \delta}$$

$$\delta'(s) = -\frac{\partial F(x(s),\delta(s))}{\partial x}$$

  * The idea of the algorithm is to construct a path from $s = 0$ that traces the correspondence of all pairs $(x(s), \delta(s))$ until $\delta = 1$. All we need is a starting point.
  * In this example, the starting point can be $\delta(0) = 0$ and $x(0) = 0.5$
  * Then, for $s' = s + ds$, we can follow the path traced by the differential equation:

$$x(s') = x(s) + x'(s) = x(s) + \frac{\partial F(x(s),\delta(s))}{\partial \delta}$$

$$\delta(s') = \delta(s) + \delta'(s) = \delta(s) - \frac{\partial F(x(s),\delta(s))}{\partial x}$$

– How does it apply to MPEs?

  * Need to start the homotopy at a starting point (i.e. one arbitrary MPE).
  * How? Follow P-M algorithm for alternative starting values.
  * For a given starting point, iterate on the following multi-dimention differential equation:

$$y_i'(s) = (-1)^{i+1} \det \left(\left(\frac{\partial F(y(s); \rho)}{\partial y} \right)_{-i} \right)$$

    where $y(s) = (V(s), p(s), \delta(s))$, and $i = 1, ..., 2M^2 + 1$.
  * In this particular example, Proposition 2 shows that the MPE is unique for $\delta = 0$ or $\delta = 1$ (i.e. no or full forgetting). This is a natural starting point to trace the "main" correspondence.
  * Finding all the equilibria is not guaranteed however, since some MPEs can be disconnected from the main path. Need to start the algorithm at arbitrary points in the interior.

– What causes the multiplicity?

  * A **sufficient** condition for uniqueness of MPE is: (i) statewise uniqueness (i.e. stage game), and (ii) the movements through the state space is unidirectional.
  * Without organizational forgetting, firms with probability one end up at (M, M).
  * Working backward establish uniqueness.

– Distribution of the number of computed equilibria:

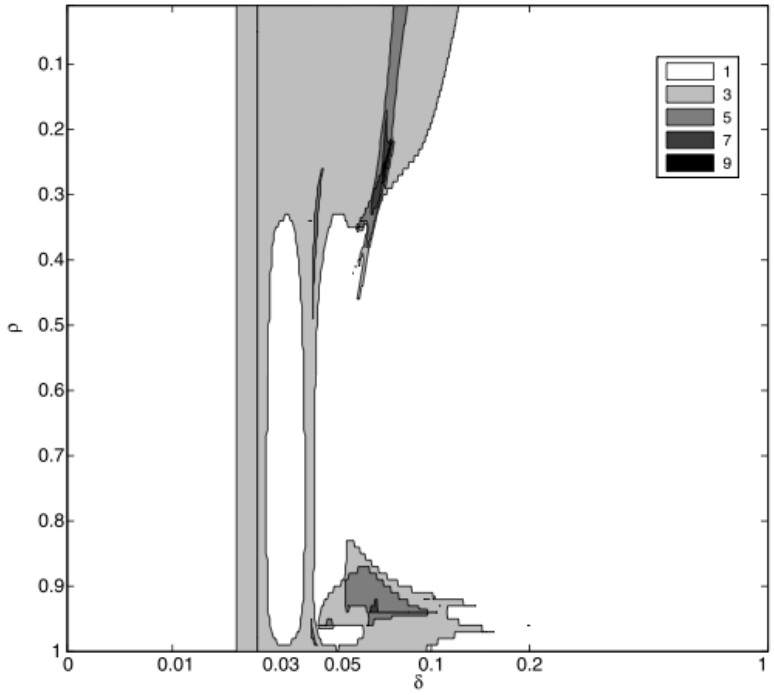


FIGURE 2.—Number of equilibria.

- Types of equilibria:
  1. Flat equilibrium without well (i.e. without forgetting)
  2. Flat equilibrium with well (i.e. small level of forgetting)
  3. Trenchy equilibrium (i.e. small level of forgetting)
  4. Extra trenchy equilibrium (i.e. large level of forgetting)

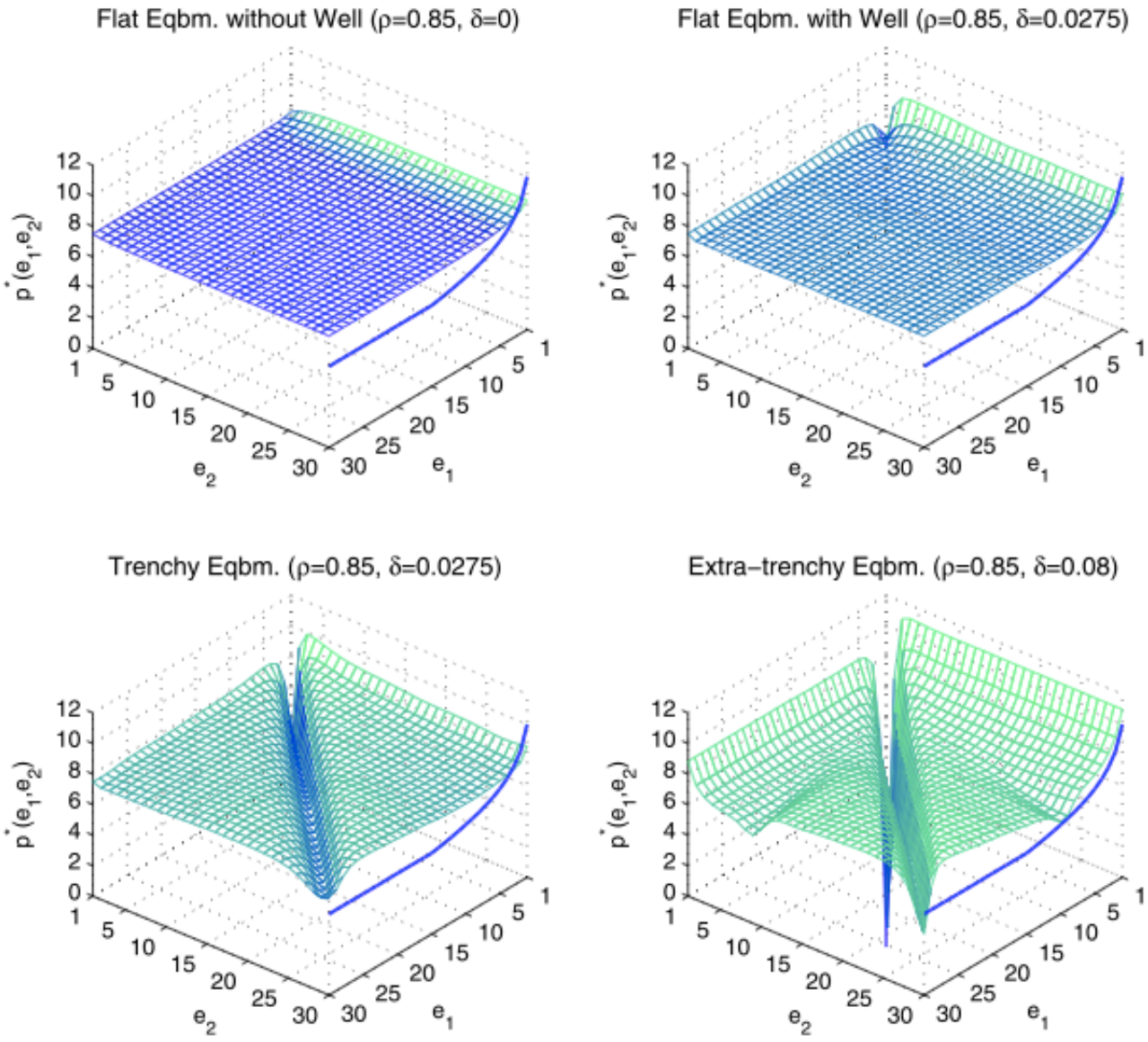


FIGURE 4.—Policy function $p^*(e_1, e_2)$; marginal cost $c(e_1)$ (solid line in $e_2 = 30$ plane).

- Industry dynamics:
  - No forgetting: Symmetric structure
  - Introducing forgetting lead to more asymmetric structures:
    - In "flat" equilibria, asymmetries are mostly temporary, and are associated with initial price wars (i.e. when $e_1 = e_2 = 1$).
    - In trenchy equilibria, asymmetries are persistent
  - The type of equilibria and the level of forgetting therefore determine the degree of concentration in the long-run, and the life cycle of "dominant" firms.

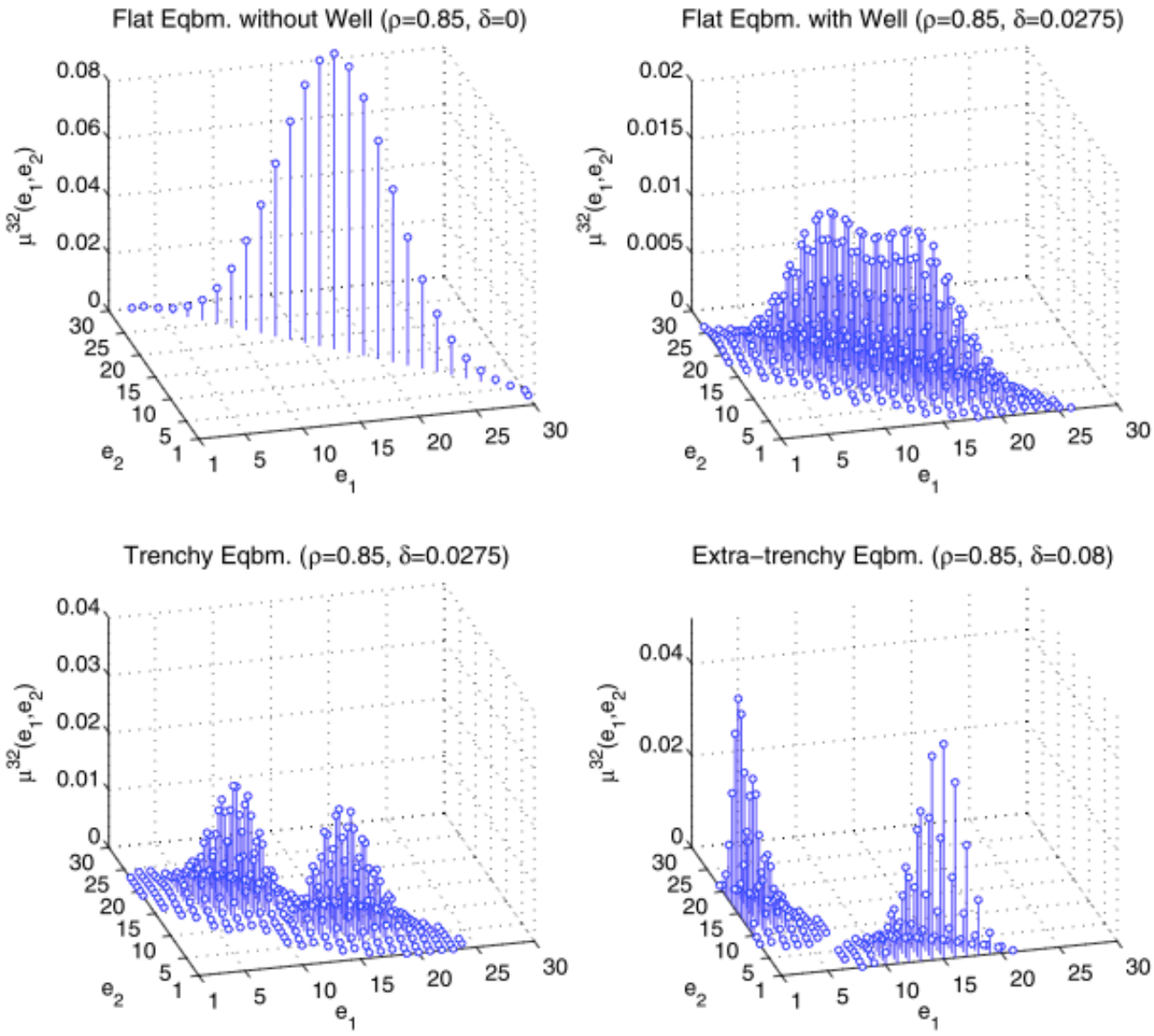


FIGURE 6.—Transient distribution over states in period 32 given initial state (1, 1).

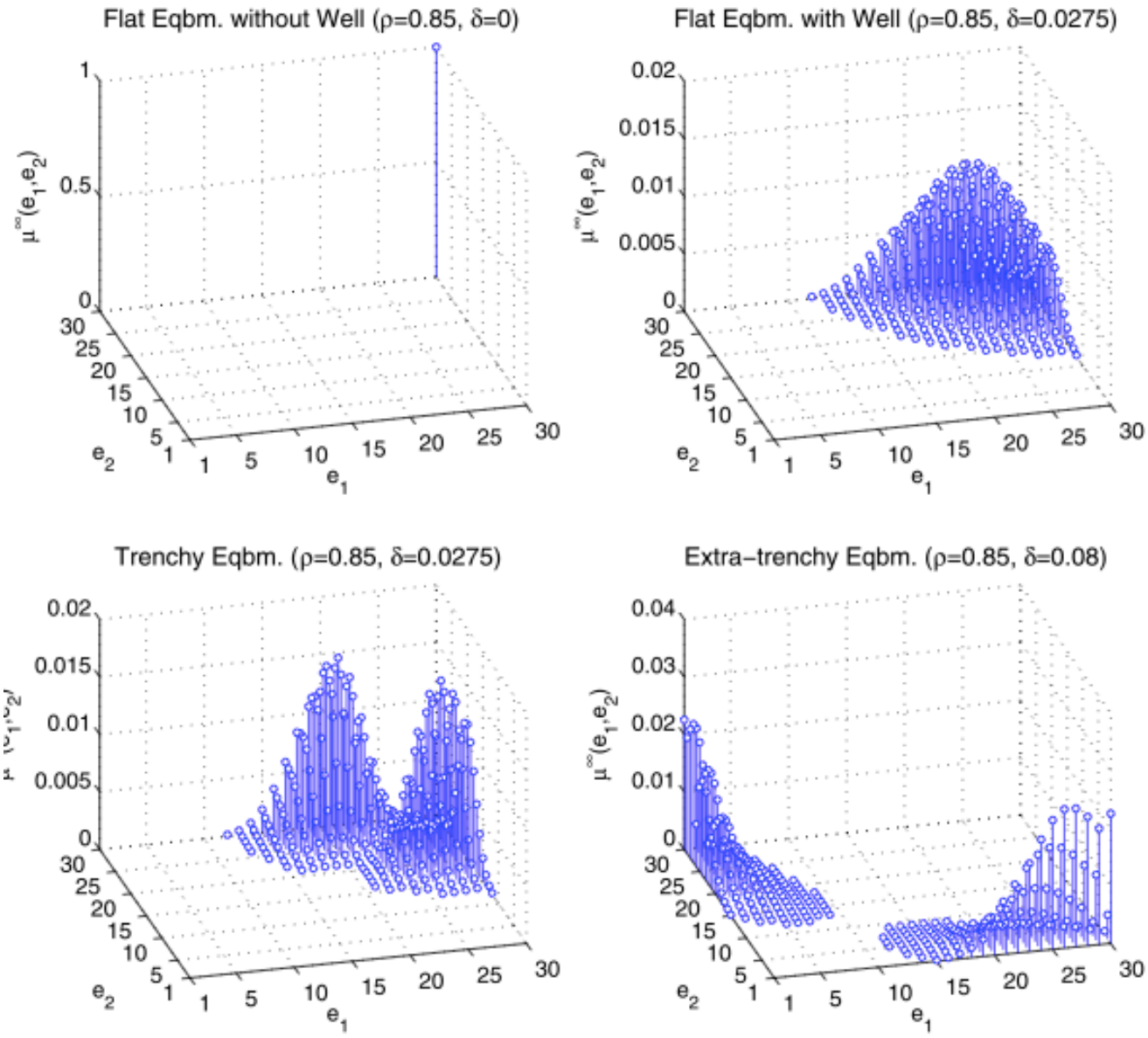


FIGURE 7.—Limiting distribution over states.

- Dynamic pricing and price wars:
  - We can re-write the FOC as follows:

$$p^*(\mathbf{e}) = c^*(\mathbf{e}) + \frac{\sigma}{1 - D_1^*(\mathbf{e})}$$

    where $c^*(\mathbf{e}) = c(e_1) - \beta \left(\bar{V}_{11}(\mathbf{e}) - \bar{V}_{12}(\mathbf{e}) \right)$.
  - **Wells VS Trenches:**
    - Both phenomena have an investment dimension:
      - Firms want to price low to acquire experience, and reach a level of dominance.
    - Trenches also have a defensive incentive:
      - Dominant firms try to maintain their dominance position as soon as their opponent gets closer to 1/2 share.
  - **Sideway trenches:**
    - With "extra trenchy" equilibria, price wars occur even with asymmetric market structures.
    - Why? When asymmetries are large, followers with very low experience face a larger probability of loosing experience, than making a sale (i.e. gaining e). Therefore, dominant firms can "invest" in their dominance position by increasing the probability that the followers stays in that region.

Summary of Predictions

PRICING BEHAVIOR AND INDUSTRY DYNAMICS

	Flat Eqbm. Without Well	Flat Eqbm. With Well	Trenchy Eqbm.	Extratrenchy Eqbm.
Leading example	$\rho = 0.85$, $\delta = 0$	$\rho = 0.85$, $\delta = 0.0275$	$\rho = 0.85$, $\delta = 0.0275$	$\rho = 0.85$, $\delta = 0.08$
Preemption battle (well)	no	yes	no	no
Price war triggered by imminent threat (diagonal trench)	no	no	yes	yes
Price war triggered by distant threat (sideways trench)	no	no	no	yes
Short-run market dominance	no	yes	yes	yes
Long-run market dominance	no	no	yes, modest	yes, extreme
Dominance properties	yes	no, mostly	no, mostly	no, mostly

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