Lecture 3: Aggregate demand models

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Introduction: Omitted attributes

- A motivating example: Trajtenberg (1989)
- **Research question:** What is the value of technology innovation?
  - *Case study:* Diffusion of CT scans
- **Demand model:**

$$\sigma_{jt} = \frac{\exp(g(x_{jt}) + \alpha(y_i - p_{jt}))}{\sum_{j' \in J_t} \exp(g(x_{jt}) + \alpha(y_i - p_{jt}))} = \frac{\exp(g(x_{jt}) - \alpha p_{jt}))}{\sum_{j' \in J_t} \exp(g(x_{j't}) - \alpha p_{j't})}$$

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  where z_{jt} is a vector characteristics and p_{jt} is the "residual" price of j (hedonic).
- Value of technological progress:

$$\begin{aligned} \Delta W_t &= \frac{1}{\hat{\alpha}} \times \left[\ln \left(\sum_{j \in J_t} \exp(\hat{g}(x_{jt}) - \hat{\alpha} p_{jt}) \right) \right. \\ &\quad \left. - \ln \left(\sum_{j \in J_{t-1}} \exp(\hat{g}(x_{j,t-1}) - \hat{\alpha} p_{j,t-1}) \right) \right] \end{aligned}$$

Introduction: Omitted attributes

	1976	1977	1978	1979	1980	1981
RPRICE	11.252	.993	1.020	.485	.695	−.277
	(6.4)	(4.8)	(4.8)	(1.8)	(2.4)	(−2.5)
SPEED	−2.292	2.138	4.624	−8.669	11.347	−7.504
	(−7.3)	(2.8)	(1.0)	(−1.5)	(2.0)	(−.5)
$SPEED^2$	.236	−1.264	−8.283	31.292	−34.838	74.161
	(4.0)	(−3.4)	(−.6)	(1.9)	(−1.6)	(1.4)
RESOL	69.107	9.113	−34.126	−15.283	−18.129	32.877
	(7.3)	(2.4)	(−6.3)	(−5.0)	(−3.6)	(−3.9)
$RESOL^2$	−23.360	−2.533	15.096	6.291	7.738	−24.028
	(−7.6)	(−1.5)	(5.8)	(3.8)	(2.7)	(−4.2)
RTIME	−3.931	5.082	2.385	3.288	3.161	−2.591
	(−5.3)	(7.0)	(2.0)	(3.3)	(2.8)	(−2.8)
$RTIME^2$	1.054	−2.370	−1.511	−1.401	−2.093	5.560
	(4.5)	(−6.7)	(−2.0)	(−2.1)	(−2.2)	(3.9)
$\rho^2 = 1 - [L(\beta^*)/L(\beta^0)]$	.29	.12	.16	.16	.20	.14
$Corr(\pi^*, \pi)$	.999	.877	.900	.870	.722	.547
	(.0001)	(.0001)	(.0001)	(.0001)	(.0024)	(.082)
Number of scanners	8	15	16	16	15	11
Number of observations	285	324	164	177	193	153

NOTE.—Asymptotic *t*-values are in parentheses.

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- Upward sloping demand!
- **Omitted variable bias:** Unobserved quality of new scanners

Data: Aggregate demand and characteristics

- **Data:** Panel of market shares and product characteristics:

$$\{s_t, p_t, x_t\}_{t=1,\ldots,T}$$

  where t indexes a market, $x_t = \{x_{jt,1}, \ldots, x_{jt,K}\}_{j=1,\ldots,n_t}$ is a matrix of observed characteristics, and $\{p_t, s_t\} = \{p_{jt}, s_{jt}\}_{j=1,\ldots,n_t}$ is a matrix of endogenous prices and market shares.
- s_{jt} denotes the observed share of consumers choosing j in market t (e.g. geography, time period, etc)
- Measurement?

$$s_{jt} = \frac{Q_{jt}}{M_t}$$

  where M_t is the number of potential buyers.
- I will use w_{jt} to denote a vector of price instruments.

Incorporating Omitted Attributes: Logit case

- Relaxing the conditional independence assumption (Berry 1994):

$$V_{ijt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt} + \epsilon_{ijt}$$

  where ξ_{jt} is unobserved by the econometrician.
- This is a clear violations of the assumption and residuals are independent of (x, p):

$$E(\xi_{jt} + \epsilon_{ijt} | x_{jt}, p_{jt}) \neq 0$$

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- MLE can identify (at most) a product/market fixed-effect:

$$\delta_{jt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$$

  Normalization: $\delta_{0t} = 0$.
- How?

$$\hat{\delta}_{jt} = \sigma_{jt}^{-1}(s_t) = \ln s_{jt} - \ln s_{0t} = \text{Log odds ratio}$$

Omitted attributes: Instrumental Variables

- Identification of (β, α)? Instrumental variables.

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- **Assumptions:**
  - Exogenous characteristics: $E[\xi_{jt}|x_t] = 0$
  - Price instrument (e.g. cost shifter): $E[\xi_{jt}|z_{jt}] = 0$
- Notation:
  - $y_{jt} = \ln s_{jt} - \ln s_{0t}$
  - Matrix of characteristics: Y, X and Z (X includes price)

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- Estimation: GMM

$$\text{Moment condition: } E\left[\xi_{jt} \cdot z_{jt}\right] = 0$$

$$\text{Empirical moments: } \frac{1}{n}\sum_{j,t} \underbrace{\left(y_{jt} - x_{jt}\beta + \alpha p_{jt}\right) \cdot z_{jt}}_{g_{jt}(\theta)} = 0$$

  where $n = \sum_t J_t$ is the number of observations and $\theta = (\alpha, \beta)$
- Empirical moments in matrix form:

$$\bar{g}_n(\theta) = n^{-1} Z'(Y - X\theta)$$

Omitted attributes with Non-IIA Demand: Nested-Logit

Source: Berry (1994)

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- **Example:**
  - G segments

    $$V_{ijt} = \delta_{jt} + \nu_{ig} + (1-\lambda)\epsilon_{ijt} \text{ If } j \in g$$

    λ is nested-logit parameter (i.e. correlation in errors).
  - Demand:

    $$\sigma_j(\delta_t, G_t) = \frac{\exp(\delta_{jt}/(1-\lambda))}{H_g \left[\sum_{g'} H_{g'}\right]}$$

    where $H_g = \ln\left[\sum_{j \in g} \exp(\delta_{jt}/(1-\lambda))\right]$ is the inclusive-value of segment g.

Omitted attributes with Non-IIA Demand: Nested-Logit

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- Inverse-demand:

$$s_{jt} = \sigma_j(\delta_t, G_t; \lambda) \rightarrow \delta_{jt} = \ln s_{jt} - \ln s_{0t} - \lambda s_{j|g,t} = \sigma_j^{-1}(s_t, G_t; \lambda)$$

where $s_{j|g,t}$ is the conditional share of j in segment g.

Nested-Logit

Source: Berry (1994)

- **Implication:** The inverse demand takes the form of a linear regression

$$y_{jt} = \ln s_{jt}/s_{0t} = x_{jt}\beta + \lambda s_{j|g,t} + \xi_{jt}$$

- **Implication:** $(\beta^{ols}, \lambda^{ols})$ is biased. Why?
  - The popularity of j in segment g is function of ξ_{jt}
  - $\hat{\lambda}^{ols}$ is biased upward: Too much correlation in taste
  - This biased is present wether or not characteristics are exogenous

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  - This biased is present wether or not characteristics are exogenous
- **Omitted variable bias:** When a product is popular in a segment. Two possible reasons:
  - **Large** λ**:** Rival products have unfavorable attributes (or fewer options)
  - **Zero** λ**:** Product j has large *unobserved* quality
- Without further assumptions on (G_t, x_t) it impossible consistently identify λ

Nested-Logit

Source: Berry (1994)

- **Assumption:** The residual quality of j is independent of the menu of characteristics $x_t = \{x_{1t}, \ldots, x_{J_t,t}\}$ available in market t

$$E\left[\xi_{jt} | x_t\right] = 0$$

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- This conditional moment restriction (CMR) was introduced by Berry (1994) and Berry et al. (1995)
- This is not as strong as it seems like...
- The quality index δ_{jt} is linear in ξ_{jt} and so we can condition of rich fixed-effects. E.g.:

$$\delta_{jt} = x_{jt}\beta + \mu_j + \tau_t + \xi_{jt} \rightarrow E\left[\xi_{jt}|x_t, \mu_j, \tau_t\right] = 0$$

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- **Example:** Moment conditions with option fixed-effects:

$$E\left[\xi_{jt} \cdot \dot{z}_{jt}\right] = E\left[\left(\dot{y}_{jt} - \dot{x}_{jt}\beta - \lambda \ln \dot{s}_{j|g,t}\right) \cdot \dot{z}_{jt}\right]$$

where $\dot{x}_{jt} = x_{jt} - \bar{x}_j$ is the "within-transformation" of characteristics.

Nested-Logit

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- Examples of relevant IVs for λ:
  - Number of products in the same nest
  - Characteristics of products in the same nest

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- Examples of relevant IVs for λ:
  - Number of products in the same nest
  - Characteristics of products in the same nest
- Adding back prices:

$$y_{jt} = \ln s_{jt}/s_{0t} = x_{jt}\beta - \alpha p_{jt} + \lambda s_{j|g,t} + \xi_{jt}$$

- **Takeaway:** We need two independent sources of exogenous variation
  - IV for p: Cost/markup shifter
  - IV for s: Characteristics of rival products

General Case: Mixed-logit demand system

- Indirect utility function (Nevo 2001):

$$V_{ijt} = \begin{cases} \sum_{k=1}^{K} \beta_{i,k} x_{jt,k} - \alpha_i p_{jt} + \xi_{jt} + \epsilon_{ij} & \text{If } j \neq 0 \\ \epsilon_{i0} & \text{Else.} \end{cases}$$

$$\beta_{i,k} = \beta_k + z_i \pi_k + \nu_{i,k}$$

$$\alpha_i = \alpha + z_i \pi_p + \nu_{i,p}$$

$z_i \sim D_t(\cdot)$ (known) and $\nu_i \sim F(\nu_i; \lambda)$ (unknown)

Average utility: $\delta_{jt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$

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- Characteristics vector: $x_{jt} = \left\{ x_{jt}^{(1)}, x_{jt}^{(2)} \right\}$.
  - Attributes with common valuation: $x_{jt}^{(1)}$
  - Attributes with heterogenous valuation: $x_{jt}^{(2)}$
- Parameters:
  - *Linear:* Mean utility parameters (β, α)
  - *Non-linear:* Demographic weights (π) and random-coefficients (λ)

Baseline Model: Exogenous Characteristics

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where $\delta_{jt} = \beta_0 + x_{jt}^{(1)}\beta_1 + x_{jt}^{(2)}\beta_2 + \xi_{jt}$.

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- The **residual** of the model is obtained from the inverse-demand function:

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- **Important:** The model residual is a non-linear function of λ. This leads to a non-linear IV estimator

Identifying Assumption

- **Assumption:** The unobserved attribute of each product is independent of the **menu**, x_t, of characteristics available in market t,

$$E[\xi_{jt}|x_t] = 0 \quad \text{(CMR)}.$$

- In practice, the model is estimated using a finite number (L) of unconditional moment restrictions, $A_j(x_t)$ (aka instruments):

$$\begin{aligned} & E\left[\rho_j(s_t, x_t; \theta^0) \cdot A_j(x_t)\right] = 0 \\ \leftrightarrow \quad & E\left[\left(\sigma_j^{-1}\left(s_t, x_t^{(2)}; \lambda^0\right) - x_{jt}\beta\right) \cdot A_j(x_t)\right] = 0. \end{aligned}$$

- **Questions:**
  - How to choose $A_j(x_t)$?
  - How to estimate (β, λ)?

How should we choose the instruments?

- True model: Random-coefficient with exogenous attributes

$$\begin{aligned}\sigma_j(\delta_t, x_t^{(2)}|\lambda) &= \int \frac{\exp(\delta_{jt} + \lambda\eta_i x_{jt}^{(2)})}{1 + \sum_{j'} \exp(\delta_{j't} + \lambda\eta_i x_{j't}^{(2)})} dF(\eta_i|\lambda) \\ \Rightarrow \quad & x_{jt}\beta + \xi_{jt} \equiv \delta_{jt} = \sigma_j^{-1}(s_t, x_t^{(2)}|\lambda)\end{aligned}$$

- Wrong model: Logit

$$\sigma_j^{-1}\left(s_t, x_t^{(2)}|\lambda = 0\right) = \ln s_{jt}/s_{0t} = x_{jt}\beta + \underbrace{\Delta_{jt} + \xi_{jt}}_{r_{jt}=\text{Logit residual}}$$

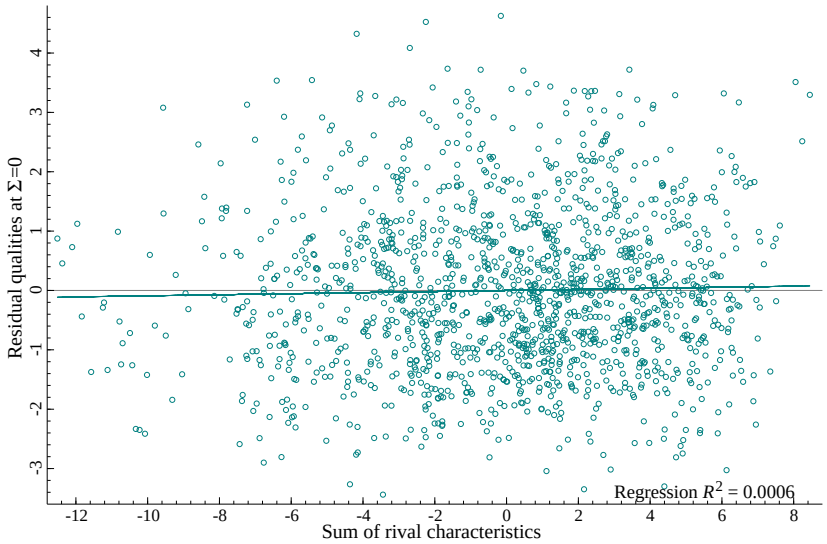
  where $\Delta_{jt} = \sigma_j^{-1}(s_t, x_t^{(2)}|\lambda = 0) - \sigma_j^{-1}(s_t, x_t^{(2)}|\lambda)$ (omitted variable)
- Identification: $A_j(x_t)$ is a strong and valid instrument if

$$E[\xi_{jt}A_j(x_t)] = 0 \text{ and } E[r_{jt}A_j(x_t)] \neq 0$$

Identification in a Picture

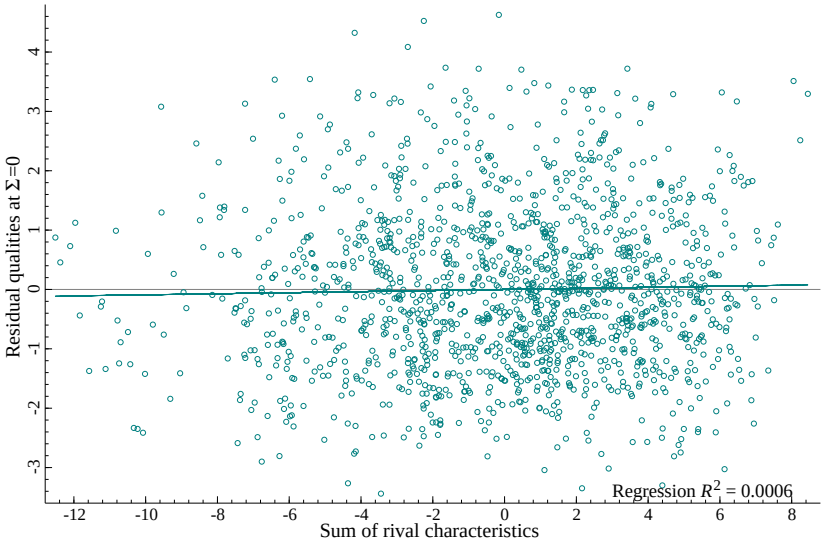
Identification in a Picture

(A) IV: Sum of rivals' characteristics

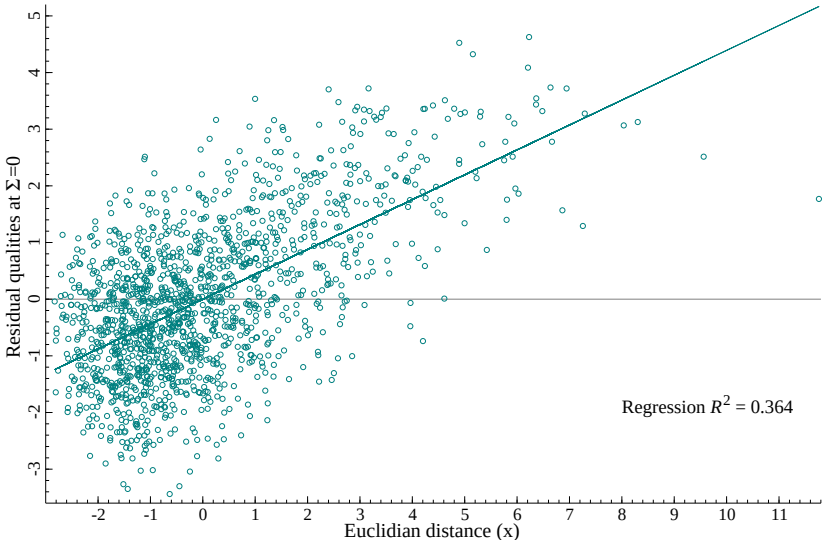


Identification in a Picture

(A) IV: Sum of rivals' characteristics



(B) IV: Euclidean distance in x



- **Interpretation:** Products that have few close substitutes ($\uparrow$ distance) have large market-share (at the true λ)
- Under logit ($\lambda = 0$), isolated products are predicted to have higher quality (i.e. $\uparrow \Delta_{jt}$)
- Clear violation of the moment condition

Instrument selection: General result

- An instrument function is "strong" if it is a good predictor of the inverse-demand function:

$$E[\sigma^{-1}(s_t, x^{(2)}|\lambda)|x_t] \approx A_j(x_t)\gamma$$

This corresponds to the reduced-form of the model.

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- **Proposition (Gandhi and Houde 2020):** If the distribution of $\{\xi_j\}_{j=1,\ldots,n}$ is exchangeable (conditional on x_{jt}), then the reduced form can be written as

$$E\left[\sigma_j^{-1}\left(\boldsymbol{s}, \boldsymbol{x}^{(2)}; \boldsymbol{\lambda}^0\right)|\boldsymbol{x}\right] = g\left(\boldsymbol{d}_j\right)$$

  where g is a **symmetric** function of the state vector.

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  where g is a **symmetric** function of the state vector.

- **Implication:** The optimal instrument is a function of the distribution of characteristics differences (aka Differentiation IVs)

Why is symmetry of the reduced-form useful?

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- **Definition:** $g(d_1, d_2)$ is a symmetric function if $g(d_1, d_2) = g(d_2, d_1)$.

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- **Definition:** $g(d_1, d_2)$ is a symmetric function if $g(d_1, d_2) = g(d_2, d_1)$.
- **Example:** Single dimension $d_{jt} = \{x_{1t} - x_{jt}, x_{2t} - x_{jt}, \ldots, x_{J_t,t} - x_{jt}\}$
  - Second-order approximation of $g(d)$:

$$\begin{aligned} g(d_{jt}) &\approx \sum_{j'} \gamma^1_{j'} d_{jt,j'} + \sum_{j'} \gamma^2_{j'} (d_{jt,j'})^2 + \gamma^3 \left(\sum_{j'} d_{jt,j'}\right)^2 \\ &= \gamma^1 \left(\sum_{j'} d_{jt,j'}\right) + \gamma^2 \left(\sum_{j'} (d_{jt,j'})^2\right) + \gamma^3 \left(\sum_{j'} d_{jt,j'}\right)^2 \end{aligned}$$

Implication 1: Polynomial Basis

- Single dimension measures of differentiation

$$\text{Quadratic: } A_j(x_t) = \sum_{j'} \left(d^k_{jt,j'} \right)^2$$

  Note: $\sqrt{z_{jt,k}}$ is the Euclidian distance between product j and its rivals in market t along dimension k.

- Adding interaction terms:

$$\text{Covariance: } A_j(x_t) = \sum_{j'} d^k_{jt,j'} \times d^l_{jt,j'}$$

Implication 2: Histogram Basis

- **Note:** This approach is advisable only in very large samples (+large choice-sets), and when the goal is to estimate a very flexible distribution of RCs.
- Single dimension measure of differentiation = Number of rivals in discrete bins

$$A_j(x_t) = \left\{ \sum_{j'} 1\left(d^k_{jt,j'} < \kappa_l \right) \right\}_{l=1,\ldots,L}$$

- Multi-dimension measure of differentiation:

$$A_j(x_t) = \left\{ \sum_{j'} 1\left(d^k_{jt,j'} < \kappa_l \right) 1\left(d^{k'}_{jt,j'} < \kappa_{l'} \right) \right\}_{l=1,\ldots,L, l'=1,\ldots,L}$$

Implication 3: Local Basis

- **Note:** In most parametric models, the inverse demand is function of characteristics of **close-by** rivals. Therefore, in the previous histogram basis, we should be focussing on "local" rivals.
- Single dimension measure of differentiation = Number of nearby rivals along each dimension

$$A_j(x_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right), \text{ e.g. } \kappa_k = sd(x_{jt,k})$$

- Multi-dimension measure of differentiation:

$$A_j(x_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times d_{jt,l}, \text{ e.g. } \kappa_k = sd(x_{jt,k})$$

- When $x_{jt,k}$ is discrete, this basis function boils down to the familiar Nested-logit IVs.
  - Number of competitors and characteristics of rivals within segment

Differentiation IVs with Endogenous Prices

Example with cost shifter

1. First-stage price regression:

$$\hat{p}_{jt} = \hat{\pi}_0 + \hat{\pi}_1 x_{jt} + \hat{\pi}_2 \omega_{jt}$$

   This could be richer/more non-linear. Goal: Provide a good fit using (x, w)

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2. Differentiation IV: Quadratic

$$\sum_{j'} \left(d^{\hat{p}}_{jt,j'} \right)^2 \text{ and } \sum_{j'} \left(d^{\hat{p}}_{jt,j'} \right)^2 \cdot \boldsymbol{d}_{jt,j'}$$

   where $\boldsymbol{d}_{jt,j'} = (d^{x}_{jt,j'}, d^{\hat{p}}_{jt,j'})$.

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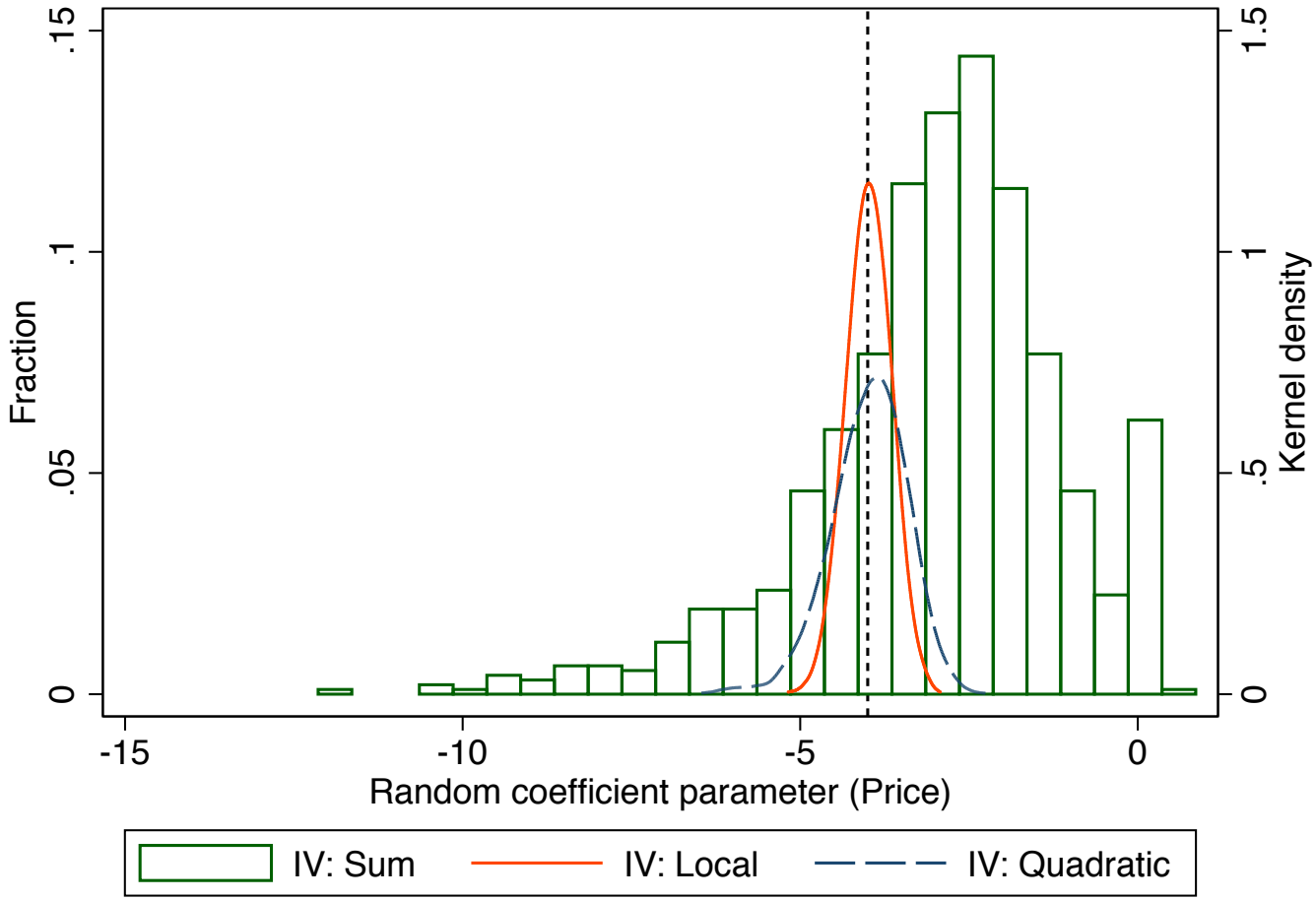
   where $\boldsymbol{d}_{jt,j'} = (d^{x}_{jt,j'}, d^{\hat{p}}_{jt,j'})$.

3. Differentiation IV: Local

$$\sum_{j'} \left(|d^{\hat{p}}_{jt,j'}| < \text{sd}(\hat{p}_{jt}) \right) \text{ and } \sum_{j'} \left(|d^{\hat{p}}_{jt,j'}| < \text{sd}(\hat{p}_{jt}) \right) \cdot \boldsymbol{d}_{jt,j'}$$

Does it matter?

Monte-carlo simulation results from the quality-ladder model



Dash vertical line = True parameter value

How can we estimate the model?

- Non-linear GMM estimator:
  - Moment conditions:

$$\bar{g}(\beta,\theta) = \frac{1}{n}\sum_{j,t} Z_{jt} \underbrace{\left(\sigma_j^{-1}(s_t, x_t^{(2)}, p_t|\lambda) - X_{jt}\beta\right)}_{\text{Residual: } \rho_{jt}(\beta,\lambda)} = 0$$

  - Nested fixed-point optimization problem:

$$\begin{aligned} \min_{\lambda} \quad & \rho(\lambda)^T Z W^{-1} Z^T \rho(\lambda) \\ s.t. \quad & \rho_{jt}(\lambda) = \sigma_j^{-1}(s_t, x_t^{(2)}, p_t|\lambda) - X_{jt}\beta(\lambda) \\ & \beta(\lambda) = \text{Linear IV estimate given } \lambda \end{aligned}$$

  - Note: Expressing β as a function of λ greatly simplifies the optimization problem (i.e. $dim(\lambda) << dim(\beta)$).

Three computational challenges

- Numerical optimization problem: BFGS or Simplex (relatively simple when IVs are strong)
- Numerical integration:

$$\begin{aligned}\sigma_j(\delta_t, x_t^{(2)}, p_t|\lambda) &= \int \frac{\exp(\delta_{jt} + \alpha_i p_{jt} + x^{(2)}\nu_i)}{1 + \sum_{j'} \exp(\delta_{j',t} + \alpha_i p_{j',t} + x_{j',t}^{(2)}\nu_i)} dF(\alpha_i, \nu_i|\lambda) \\ &\approx \sum_r w_r \frac{\exp(\delta_{jt} + \alpha_r p_{jt} + x^{(2)}\nu_r)}{1 + \sum_{j'} \exp(\delta_{j',t} + \alpha_r p_{j',t} + x_{j',t}^{(2)}\nu_r)}\end{aligned}$$

  Example: Monte-carlo simulation ($w_r = 1/R$), quadrature method ($w_r =$ gaussian weights).
- Non-linear equation solution (fixed-point):

$$\delta_{jt} = \sigma_j^{-1}(s_t, x_t^{(2)}, p_t|\lambda)$$

Pseudo-code

```
/* GMM objective function */
gmm_obj(const vP, const adFunc, const avScore, const amHessian)
{
  /* Invert demand */
  inverse(&vDelta0,vP);
  /* Quality decomposition */
  decl vWithinDelta0=within(vDelta0,vFEid);
  decl vLParam=ivreg(vWithinDelta0,mX,mIV,A);
  decl vXi=vWithinDelta0-mX*vLParam;

  /* GMM objective function */
  mG=vXi.*mIV;
  decl scale=100;
  decl g=sumc(mG);
  if(isnan(vDelta0)==1) adFunc[0]=.NaN;
  else adFunc[0]=double(-g*A*g'/scale);
  if(avScore) {
    /* Score */
    decl mJacobian;
    jacobian(&mJacobian,vDelta0,vP);
    mJacobian=within(mJacobian,vFEid);
    decl dG=(mMx*mJacobian)'mIV;
    decl vScore=-2*dG*A*g'/scale;
    avScore[0]=vScore;
  }
  return 1;
}
```

Demand inversion: Fixed-point algorithm

- **Algorithm 1 (Berry et al. (1995)):** The following function is a contraction mapping

$$\delta_{jt}^{i} = \Gamma(\delta_t^{i-1}) \equiv \delta_{jt}^{-1} + \underbrace{\left(\ln s_{jt} - \ln \sigma_j(\delta_t^{i-1})\right)}_{\text{step size}}$$

1. Initial guess: δ_{jt}^{0}
2. Iteration i:
$$\delta_{jt}^{i} = \delta_{jt}^{i-1} + \left(\ln s_{jt} - \ln \sigma_j(\delta_t^{i-1})\right)$$
3. Stop if $||\ln s_{jt} - \ln \sigma_j(\delta_t^{i-1})|| < \varepsilon$. Else repeat step 2.

Demand inversion: Fixed-point algorithm

- **Algorithm 2:** Newton's nonlinear root-finding algorithm

- Univariate example:

$$f(x) = 0$$

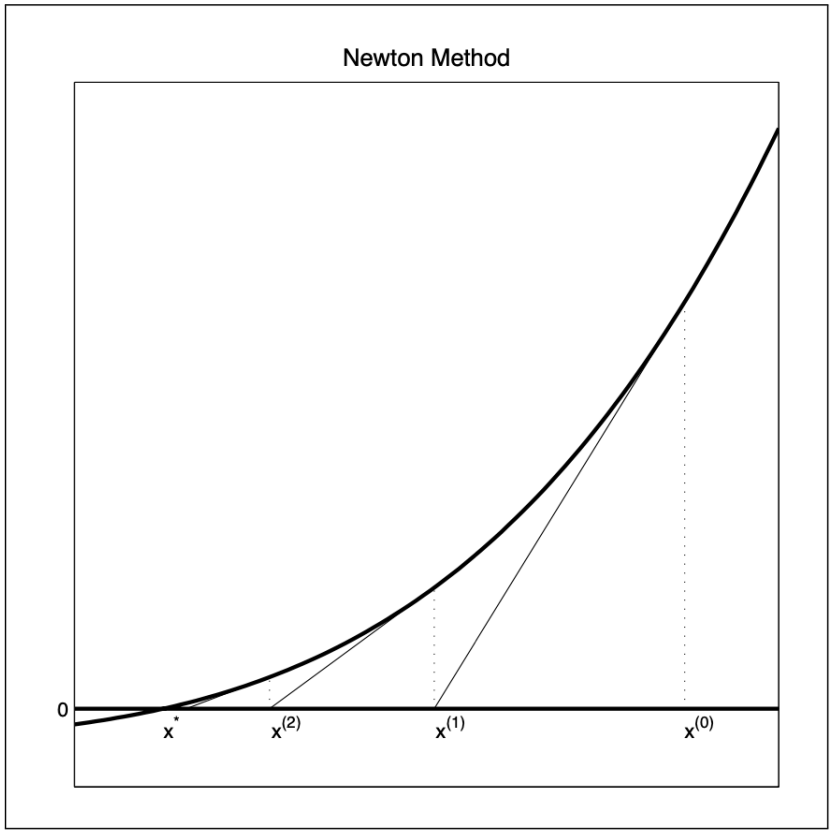
- Initial guess: x^0
- Linear approximation:

$$0 = f(x) \approx f(x^0) + f'(x^0)(x^1 - x^0)$$

$$x^1 = x^0 - f'(x^0)f(x^0)$$

- Iteration k:

$$x^k = x^k - f'(x^k)f(x^k)$$



Demand inversion: Fixed-point algorithm

- **Algorithm 2:** Newton's nonlinear root-finding algorithm
  - "Zero" functions ($J_t \times 1$):

$$f(\delta_t) = \ln s_t - \ln \sigma(\delta_t)$$

  - Jacobian matrix:

$$Df(\delta) = -\left[\frac{\partial \sigma_j(\delta_t)}{\partial \delta_{kt}}\right] / \sigma(\delta_t)$$

    where

$$\frac{\partial \sigma_j(\delta_t)}{\partial \delta_{kt}} = \begin{cases} \sum_r w_r \sigma_j(\delta_t|\nu_r)(1 - \sigma_j(\delta_t|\nu_r)) & \text{If } j = k \\ -\sum_r w_r \sigma_j(\delta_t|\nu_r)\sigma_k(\delta_t|\nu_r) & \text{If } j \neq k \end{cases}$$

  - Pseudo-code:
    1. Initial step: $\delta_t^0 = \ln s_t - \ln s_{0,t}$ (Logit)
    2. Iteration k: Demand and Jacobian calculation

$$\sigma_{jt}(\delta_t^k) \text{ and } Df(\delta^k)$$

    3. Updating step:

$$\delta^{k+1} = \delta^k - Df(\delta)^{-1} f(\delta^k)$$

    4. Stopping rule: $f(\delta^k) < \varepsilon$.

Sample code: Demand and Jacobian calculation

```
value(const aMu,const vParam,const t)
{
  decl i;
  decl rowid=aProductID[t];
  decl mMu=new matrix[rows(rowid)][rows(aDemoID[t])];
  for(i=0;i<columns(mZ);i++) mMu+=vParam[i]*(aZ[t])[i];
  aMu[0]=exp(mMu);
  return 1;
}
demand(const mMu,const aShare,const aJac,const vDelta,const t,const vParam)
{
  decl i;
  decl rowid=aProductID[t];
  decl eV=exp(vDelta[rowid]).*mMu;
  decl mS=eV./(1+sumc(eV));
  decl vShat=meanr(mS);
  if(aJac[0]) {
    decl mD=diag(meanr(mS.*(1-mS)))-setdiagonal(mS*mS'/rows(aDemoID[t]),0);
    aJac[0]=mD;
    }
  aShare[0]=vShat;
  return 1;
}
```

Side note: Parallel computing

- Many problems we have discussed so far can be computed efficiently in parallel using multiple cores on your computer (*multithreading*), or multiple CPUs on campus (*multiprocessing*).
- When is it efficient to parallelize a problem?
  - There is no dependency between calculations that are distributed
  - Information shared across threads cannot be too large (e.g. many global variables, or very large data-sets)

Side note: Parallel computing

- **Example 1:** Life-cycle model
  - Optimization problem:

$$V_t(d,e) = \max_{c,d'} \frac{c^{1-\sigma}}{1-\sigma} + \beta EV_{t+1}(e',d')$$

$$\text{s.t.} \quad c + d' \leq (1+r)d + ew \text{ and } d' \geq 0$$

  - Backward induction:
    1. *Maximization step:* Evaluates $V_T(d_i, e_i)$ for all grid points $i = 1, \ldots, I$
    2. *Expectation step:* $EV_T(d_i, e) \approx \sum_k w_k V_T(d_i, e_k)$
    3. Repeat (1) and (2) for $t = T-1, T-2, ..., 1$.
  - Scope for parallelization?
    - ★ Cannot parallelize the calculation of V_t across time periods because processors need to know EV_{t+1}
    - ★ Given $EV_{t+1}(d,e)$, we can easily parallelize across states (e,s)
    - ★ Given $V_t(d,e)$, we can also parallelize the integration step

Side note: Parallel computing

- **Example 2:** Simulated MLE
  - Optimization problem:

$$\max_{\theta} \sum_{i=1}^{N} \ln \Pr(y_i|X_i, \theta)$$

Where $\Pr(y_i|X_i, \theta) = \sum_{r=1}^{R} w_r(\theta) \Pr(y_i|X_i, \beta_r)$

- Scope for parallelization?
  - Parallel evaluations of $\Pr(y_i|X_i, \theta)$ across individuals i (e.g. each core evaluates the choice probability for $K <<< I$ individuals)
  - Parallel evaluations of $\Pr(y_i|X_i, \beta_r)$ across simulation draws

Side note: Parallel computing

- **Example 3:** Metropolis-Hastings MCMC
  - Simulating $\theta \sim L(\cdot)$
    1. Given previous iteration θ_{i-1}: $\theta^* = \theta_{i-1} + \lambda\epsilon$ where $\epsilon \sim N(0,1)$.
    2. Evaluate likelihood at new parameter: $L(\theta^*)$
    3. Accept θ^* with probability: $\alpha = \min\left\{1, \frac{L(\theta^*)}{L(\theta_{i-1})}\right\}$
    4. Repeat (1) - (3) until convergence of the chain
  - Scope for parallelization?
    - ★ No obvious fix to parallelize the simulation step: Parallel chains violate the markov properties of the chain
    - ★ As with example 2, we can parallelize the evaluation of $L(\theta^*)$

Side note: Parallel computing

- **Example 4:** Nested fixed-point estimation
  - The "BLP" model is estimated as follows:

$$\begin{aligned} \min_{\lambda} \quad & \rho(\lambda)^T Z W^{-1} Z^T \rho(\lambda) \\ s.t. \quad & \rho_{jt}(\lambda) = \sigma_j^{-1}(s_t, x_t^{(2)}, p_t|\lambda) - X_{jt}\beta(\lambda) \\ & \beta(\lambda) = \text{Linear IV estimate given } \lambda \end{aligned}$$

  - Scope for parallelization?
    - ★ The fixed-point can be easily parallelized across markets t since demand in market t is independent of demand in market t'.

Sample code: Inversion algorithm (parallel)

```
parallel for(t=0;t<T;t++)
  {
    value(&mMu,vParam,t);
    rowid=aProductID[t];
    vIT[t]=0;
    f=1;
    do{
      if(norm(f)>eps1) {
        mJacobian=0;
        demand(mMu,&vShat,&mJacobian,vDelta,t,vParam);
      }
      else {
        mJacobian=1;
        demand(mMu,&vShat,&mJacobian,vDelta,t,vParam);
      }
      f=log(vShare[rowid])-log(vShat);
      if(mJacobian==0) vDelta[rowid]=vDelta[rowid]+f;
      else vDelta[rowid]=vDelta[rowid]+invert(mJacobian./vShat)*f;
      vIT[t]+=1;
    }while(norm(f)>eps && vIT[t]<maxit);
    if(norm(f)>eps) vDelta[rowid]=constant(.NaN,rowid);
  }
```

- Global vs Local variables?
  - *Local* variables are **not** shared across threads
  - *Global* variables **are** shared across threads
- Example:
  - The heterogenous component of utility, μ, is a local variable that takes different values across processors (t)
  - The mean utility, δ_t, is a global variable that can be modified by individual processors
  - Warning: Two processors cannot change the value of the same element of δ (rows) simultaneously.

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