# Identification and Estimation of Demand for Differentiated Products

Jean-François Houde
Cornell University & NBER

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# Starting Point: The Characteristic Approach

- **Ultimate goal:** Measure the elasticity of substitution between goods.
- **Why?**
  - Measure the degree of market power
  - Predict the effect of proposed mergers
  - Evaluate the value of new goods
  - ...
- **Natural starting point:** Linear inverse demand

$$p_{jt} = \alpha_0 + \sum_{k \in \mathcal{J}} \alpha_{j,k} q_{kt} + \epsilon_{jt}$$

- **Curse of dimensionality:** Even with *exogenous* variation in $q$'s, the number of parameters to estimate grows with the number of products.
- **Solution:** The characteristic approach (Lancaster 1966)
  - Each product is defined as a bundle of characteristics: $x_j = \{x_1, \ldots, x_k\}$
  - Consumers assigned values to each characteristics, and choose the product maximize their utility given their budget.
  - Demand for product $j$ is function of the distribution of valuations for each attribute (finite dimension).

# Discrete Choice Models

- **Two classes of models**
  - *Local:* Each product has only few close substitutes
  - *Global:* Each product is equally substitutable with all others
- **Global competition:** Logit (McFadden 1974, Anderson et al. 1992)
  - Indirect utility:

$$u_{ij} = \underbrace{\delta_j - p_j}_{\text{average}} + \underbrace{\epsilon_{ij}}_{\text{T1EV}}$$

  - Demand (shares):

$$s_j = \frac{\exp((\delta_j - p_j)/\sigma_\epsilon)}{\sum_{j' \in \mathcal{J}} \exp((\delta_{j'} - p_{j'})/\sigma_\epsilon)}$$

  - Elasticity of substitution:

$$\eta_{jk} = \left(\frac{1}{\sigma_\epsilon} s_j s_k\right) \frac{p_k}{s_j} = \frac{1}{\sigma_\epsilon} p_k s_j$$

  - Two key properties: (i) Independence of Irrelevance Alternatives (IIA), and (ii) monopolistic competition in the limit (i.e. constant markup).

# Local competition: Vertical or horizontal differentiation?

- **Definition:**
  - *Horizontal differentiation:* Consumers have different valuations of products available (even with equal prices)
  - *Vertical differentiation:* Consumers would choose the **same** product if prices were equal.
- **Horizontal Example:** Linear city model (Hotelling 1929)
  - Preferences for product $j = 1, 2$:

    $$u_{ij} = \delta - p_j - \lambda |l_i - x_j|$$

    where $l_i \sim U(0, 1)$ is the *location* (or taste) of consumer $i$, and $x_j \in (0, 1)$ is the "address" of products (i.e. characteristic) and $x_2 > x_1$.
  - Demand:

    $$s_1 = \int_0^{\bar{l}} f(l_i) dl_i = \frac{\lambda(x_2 - x_1) - p_1 + p_2}{2}$$

    where $\bar{l}$ is the "marginal" consumer (i.e. indifferent type).

# Local competition: Vertical or horizontal differentiation?

- **Vertical Example:** Quality ladder (Shaked and Sutton 1982, Bresnahan 1987)
  - Preferences for product $j = 1, 2$:

    $$u_{ij} = \theta_i x_j - p_j$$

    where $\theta_i \sim U(0, \lambda)$ is the *type* of consumer $i$, and $x_j \in (0, 1)$ is the "quality" of products (i.e. characteristic) and $x_2 > x_1$.
  - Demand:

    $$\text{Type } \bar{\theta}: \qquad \bar{\theta} x_1 - p_1 = \bar{\theta} x_2 - p_2$$

    $$s_1 = \int_0^{\bar{\theta}} f(\theta_i) d\theta_i = \bar{\theta}/\lambda = \frac{1}{\lambda} \frac{p_2 - p_1}{x_2 - x_1}$$

# Hybrid model: Random-coefficients

- The *work-horse* model is the random-utility model with heterogenous valuations for characteristics (aka random-coefficients):

$$\max_{j \in \mathcal{J}} x_j \beta_i - \alpha_i p_j + \epsilon_{ij}$$

- **Note 1:** Despite the multiplicative form, this functional form nests *some* Hotelling-style models.

$$\text{Example:} \quad u_{ij} = \delta_j - p_j - \lambda(z_j - l_i)^2 = x_j \beta_i - \alpha p_j.$$

- **Note 2:** The nested-logit model is a special case of the random-coefficient model:

$$u_{ij} = x_j \beta + \sum_k 1(j \in \mathcal{G}_k) \nu_{ik} + \epsilon_{ij}$$

where $\mathcal{G}_k$ is a nest or product-segment (i.e. discrete attribute).

# From McFadden to BLP

- **Data:** Panel of market shares and product characteristics:

$$\{s_t, p_t, x_t\}_{t=1,\ldots,T}$$

  where $t$ indexes a market, $x_t = \{x_{jt,1}, \ldots, x_{jt,K}\}_{j=1,\ldots,n_t}$ is a matrix of observed characteristics, and $\{p_t, s_t\} = \{p_{jt}, s_{jt}\}_{j=1,\ldots,n_t}$ is a matrix of endogenous prices and market shares.
- I will use $w_{jt}$ to denote a vector of price instruments (e.g. cost shifters, ownership, etc.)
- Market shares:

$$s_{jt} = q_{jt}/M_t$$

  where $M_t$ is the **observed** market size (exogenous). Examples:
    - **Cars:** U.S. population
    - **Cereals:** Population $\times$ Avg. servings per month.
    - **Gasoline:** Transportation needs (including car, bus, walk, etc.)

# From McFadden to BLP

- **Assumption:** $s_{jt}$ is measured without error (very important).
  - This often mean that you should **aggregate** data across time periods or markets to reduce the importance of measurement error in $q_{jt}$.
  - The data requirement in BLP is often underestimated... You don't need consumer-level data, but you need data on **every** consumers!
- Indirect utility function (Nevo 2001):

$$u_{ijt} = \begin{cases} \sum_{k=1}^{K} \beta_{i,k} x_{jt,k} - \alpha_i p_{jt} + \xi_{jt} + \epsilon_{ij} & \text{If } j \neq 0 \\ \epsilon_{i0} & \text{Else.} \end{cases}$$

$$\beta_{i,k} = \beta_k + z_i \pi_k + \eta_{i,k}$$

$$\alpha_i = \alpha + z_i \pi_p + \eta_{i,p}$$

$$z_i \sim F(\cdot) \text{ (known) and } \nu_i \sim N(0, \Sigma) \text{ (unknown)}$$

- Average indirect utility: $\delta_{jt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$.

# Baseline Model: Exogenous Characteristics

- Consider first a model without prices and without demographics characteristics
- **Demand:** Linear random-coefficient with T1EV random utility shocks

$$\sigma_j\left(\delta_t, x_t^{(2)}; \Sigma\right) = \int \frac{\exp\left(\delta_{jt} + \nu_i^T x_{jt}^{(2)}\right)}{1 + \sum_{j'=1}^{J_t} \exp\left(\delta_{j't} + \nu_i^T x_{j't}^{(2)}\right)} dF(\nu_i|\Sigma)$$

  where $\delta_{jt} = \beta_0 + x_{jt}^{(1)}\beta_1 + x_{jt}^{(2)}\beta_2 + \xi_{jt}$.
- The **residual** of the model is obtained from the inverse-demand function:

$$\rho_j(s_t, x_t; \theta) = \sigma_j^{-1}\left(s_t, x_t^{(2)}; \Sigma\right) - x_{jt}\beta, \quad \text{where } \theta = (\beta, \Sigma).$$

# Identifying Assumption

- **Assumption:** The unobserved attribute of each product is independent of the **menu**, $x_t$, of characteristics available in market $t$,

$$E[\xi_{jt}|x_t] = 0 \quad \text{(CMR)}.$$

- In practice, the model is estimated using a finite number ($L$) of unconditional moment restrictions, $A_j(x_t)$:

$$\begin{aligned} & E\left[\rho_j(s_t, x_t; \theta^0) \cdot A_j(x_t)\right] = 0 \\ \leftrightarrow \quad & E\left[\left(\sigma_j^{-1}\left(s_t, x_t^{(2)}; \Sigma^0\right) - x_{jt}\beta\right) \cdot A_j(x_t)\right] = 0. \end{aligned}$$

- **Gandhi & Houde (2016):** How to construct relevant instruments to identify $\Sigma$?
  - **Stock & Wright (2000)**: $A_j(x_t)$ is weak if the moment conditions are *almost* satisfied away from the true parameters.

# What in the data identifies the model?

- Like any IV problem, we need to have enough instruments to estimate $(\bar{\beta}, \Sigma)$. But what is a relevant/valid instrument?
- The CMR suggests that $\theta$ is identified by variation in the choice-set of consumers.
  - Most identification sections describe the "ideal" experiment in which a new product enters, and steal market shares from existing products with similar attributes.
- This "red-bus/blue-bus" logic is correct, but we need to use it to construct IVs, otherwise the model will be weakly identified (even if the ideal experiment is present the data-set!)

# Illustration of the Weak IV Problem

- Two detection tests:
  1. Testing the **wrong** model: IIA hypothesis

$$H_0 : E\left[\rho_j(s_t, x_t|\beta, \Sigma = 0) \cdot z_{jt}\right] = 0$$

$$\leftrightarrow \qquad \ln s_{jt}/s_{0t} = x_{jt}\beta + \gamma z_{jt} + \xi_{jt}$$
$$H_0 : \gamma = 0$$

  2. Local identification: Cragg-Donald rank test

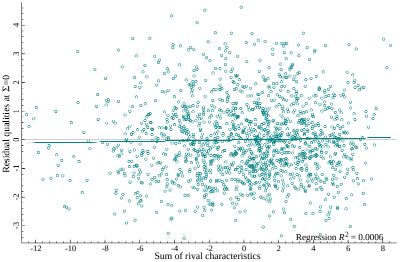
$$\text{rank}\left(E\left[\partial\rho_j\left(s_t, x_t; \theta\right)/\partial\theta^T \cdot z_{jt}\right]\right) = m$$

  This test can be implemented in STATA (ivreg2 or ranktest).

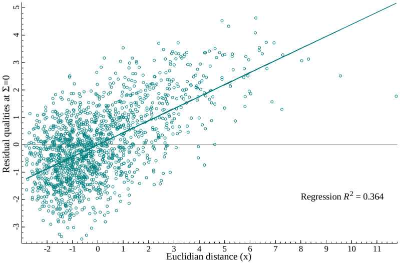
- Monte-Carlo design:
  - Sample: $T = 100$ and $J = 15$
  - Random utility with (independent) normal random-coefficients ($K_2$)
  - DGP: $(x_{jt,k}, \xi_{jt}) \sim N(0, I)$

# Weak Identification in a Picture: IIA Test

(A) IV: Sum of rivals' characteristics

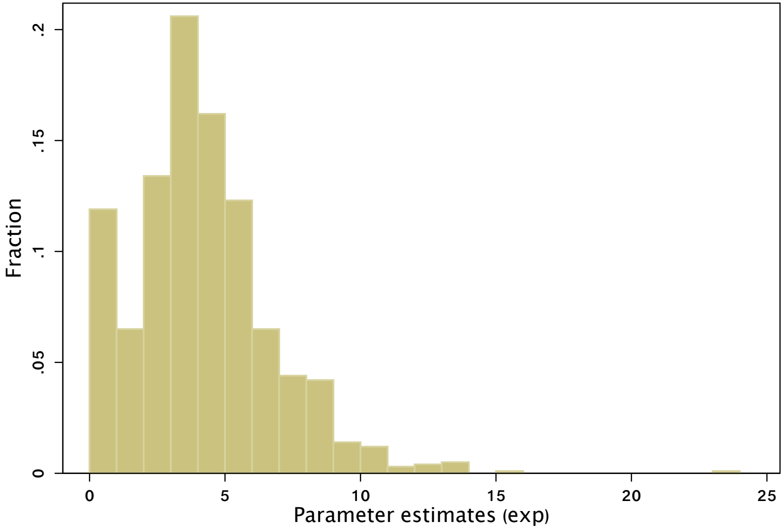



(B) IV: Euclidean distance in $x$




- **Takeaway:** Independence of $\xi_{jt}$ and the *distance of rival characteristics* rules out the IIA hypothesis, but **not** the *sum of rival characteristics*.

# Distribution of $\hat{\sigma}_2$ with weak IVs




Shapiro-Wilk test for normality: 15.71 (0). Width = 1.

# GMM Estimates with Weak IVs

| | $K_2 = 1$ | | $K_2 = 2$ | | $K_2 = 3$ | | $K_2 = 4$ | |
|---|---|---|---|---|---|---|---|---|
| | bias | rmse | bias | rmse | bias | rmse | bias | rmse |
| $\log \sigma_1$ | -11.29 | 95.93 | -5.43 | 74.95 | -1.15 | 5.50 | -8.40 | 229.67 |
| $\log \sigma_2$ | | | -4.69 | 58.31 | -1.36 | 6.26 | -1.10 | 6.17 |
| $\log \sigma_3$ | | | | | -1.41 | 9.20 | -4.66 | 112.64 |
| $\log \sigma_4$ | | | | | | | -0.93 | 4.02 |
| $\sigma_1$ | 0.14 | 2.64 | -0.01 | 2.49 | -0.03 | 2.19 | 0.22 | 2.35 |
| $\sigma_2$ | | | 0.12 | 2.42 | -0.01 | 2.27 | 0.10 | 2.30 |
| $\sigma_3$ | | | | | 0.18 | 2.38 | 0.11 | 2.38 |
| $\sigma_4$ | | | | | | | 0.08 | 2.21 |
| 1(Local-min) | 0.19 | | 0.51 | | 0.59 | | 0.66 | |
| Range(J) | 0.74 | | 1.15 | | 1.64 | | 1.51 | |
| Range(pv) | 0.17 | | 0.19 | | 0.21 | | 0.21 | |
| Range($\log \sigma$) | 11.74 | | 6.64 | | 6.58 | | 4.86 | |
| Rank-test | 1.26 | | 0.46 | | 0.26 | | 0.18 | |
| p-value | 0.62 | | 0.81 | | 0.89 | | 0.92 | |
| IIA-test | 1.33 | | 1.30 | | 1.49 | | 1.94 | |
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# Non-Parametric Identification (Berry and Haile (2014))

- To get a sense of "how" the model is identified, it is useful to consider an **ideal** setting.
  - Infinitely many markets with exogenous changes in the choice-set facing consumers.
- **Note:** BH are interested in studying the identification of the demand system $\sigma_j(x_t)$; not the distribution $f(\nu_i|\Sigma)$.
- New normalization:

$$\sigma_j^{-1}\left(s_t, x^{(2)}\right) = x_{jt}^{(1)} + \xi_{jt}$$

  where $x_{jt}^{(1)}$ is a scalar (i.e. special regressor).
- The BLP model written in this form correspond to a non-parametric instrumental variable model (Newey and Powell (2003)):

$$x_{jt}^{(1)} = \sigma_j^{-1}\left(s_t, x^{(2)}\right) + \xi_{jt}$$

# Non-Parametric Identification (Berry and Haile (2014))

- **Question 1:** Is it possible to identify $\sigma_j^{-1}(\cdot)$ from the conditional mean restriction (i.e. $E(\xi_{jt}|x_t) = 0$)?
- To answer this we need to rely on the non-parametric analog of a **rank** condition. Do we have "enough" independent excluded instruments?
- **Reduced-form:** Applying the CMR to the inverse-demand equation,

$$E[x_{jt}^{(1)}|x_t] = E\left[\sigma_j^{-1}\left(s_t, x^{(2)}\right)|x_t\right] - E[\xi_{jt}|x_t]$$

$$x_{jt}^{(1)} = E\left[\sigma_j^{-1}\left(s_t, x^{(2)}\right)|x_t\right]$$

- **Completeness condition:** For all functions $B(s_t)$ with finite expectations, if $E[B(s_t)|x_t] = 0$ almost surely, then $B(s_t) = 0$ a.s.
- If the demand function and distribution of $x_t$ satisfy this condition, then there is a unique $\sigma_j^{-1}(\cdot)$ that can be "inverted" from the reduced-form (see Theorem 1).

# Non-Parametric Identification (Berry and Haile (2014))

- **Question 2:** Is there a unique demand function associated with $\sigma_j^{-1}(s_t, x_t)$?
- The answer is yes in most mixed-logit models (see Berry (1994))
- Fore more general demand systems, Berry, Gandhi, and Haile (2013) defined a new condition that is sufficient for existence and uniqueness of an inverse demand: *connected substitutes*
  - Require that the index $\delta_{jt}$ weakly lower the market shares of all goods $k \neq j$.
  - See paper for more details...
- If both conditions are satisfied, the model is non-parametrically identified.
- The argument is only slightly more complicated with prices: The completeness condition (i.e. full rank) extends to the price IVs.

# Identification

- Back to the parametric model...
- The model is identified by the same logic:

$$E[\rho_j(s_t, x_t; \theta)|x_t] = 0, \text{ iff } \theta = \theta^0$$

$$\Leftrightarrow \underbrace{E\left[\sigma_j^{-1}\left(s_t, x_t^{(2)}; \Sigma^0\right)|x_t\right]}_{\text{Reduced-form}} - \beta_0 - x_{jt}^{(1)}\beta_1 - x_{jt}^{(2)}\beta_2 = 0$$

- **Takeaway:** The presence of a *special regressor* $x_{jt}^{(1)}$ implies that $x_{-j,t}^{(1)}$ can be used as *excluded instruments* for the endogenous shares.

# How to select the instruments?

- Since $\dim(x_t) >> dim(\Sigma) = m$, any transformation of $x_t = \{x_{1t}, \ldots, x_{J_t,t}\}$ can be used to construct valid moments:

$$E[\rho_j(s_t, x_t; \theta) \times A_j^L(x_t)] = 0, \text{ where } dim(A_j^L(x_t)) = L \geq m$$

- **Donald, Imbens, and Newey (2003):** Conditional moment restriction is equivalent to a countable number of unconditional moment restrictions (aka IVs),

$$E[\rho_j(s_t, x_t; \theta) \times A_j^L(x_t)] = 0 \text{ Iff } E[\rho_j(s_t, x_t; \theta) | x_t] = 0,$$

  where the instruments $A_j^L(x_t)$ correspond to **basis functions** spanning the space of $x_t$ (dimension $L$).
- In our context, a necessary condition for this equivalence is that the reduced-form of the model can be approximated by $A_j^L(x_t)$ as $L \rightarrow \infty$:

$$E\left[\{g_j(x_t) - A_j^L(x_t)\gamma_L\}^2\right] \rightarrow 0$$

# Curse of Dimensionality Problem

- **Curse of Dimensionality:** The reduced-form is a *product-specific* function of the entire menu of product characteristics.
  - As $J \uparrow$, both the number of arguments **and** the number of functions to approximate increase.
  - This is a *different* point than the one raised by Armstrong (2015), which is about the weakness of the price instruments as $J \to \infty$.
- Without further restrictions, we cannot directly use the insights of BH to construct relevant IVs
- **Our approach:** Reduce the dimensionality of the problem by exploiting the symmetry of the demand function (implied by the linearity of the random utility model)

# What does the characteristic structure imply for the reduced-form of the model?

- Market-structure facing product $j$ (dropping $t$):

$$(\boldsymbol{w}_j, \boldsymbol{w}_{-j}) \equiv \left(\left(\delta_j, \boldsymbol{x}_j^{(2)}\right), \left(\boldsymbol{\delta}_{-j}, \boldsymbol{x}_{-j}^{(2)}\right)\right)$$

- Properties of the linear-in-characteristics model:
  - Symmetry:
$$\sigma_j(\boldsymbol{w}_j, \boldsymbol{w}_{-j}) = \sigma_k(\boldsymbol{w}_j, \boldsymbol{w}_{-j}) \quad \forall k \neq j$$
  - Anonymity:
$$\sigma(\boldsymbol{w}_j, \boldsymbol{w}_{-j}) = \sigma\left(\boldsymbol{w}_j, \boldsymbol{w}_{\rho(-j)}\right) \forall \rho$$
  - Translation invariant: for any $\boldsymbol{c} \in \mathbb{R}^K$
$$\sigma(\boldsymbol{w}_j + (0, \boldsymbol{c}), \boldsymbol{w}_{-j} + (0, \boldsymbol{c})) = \sigma(\boldsymbol{w}_j, \boldsymbol{w}_{-j})$$

# Re-Express the Demand System

- Express the "state" of the market in *differences* relative to $j$ and treat the outside option just like any other product.
  - Characteristic differences:

$$\boldsymbol{d}_{j,k}^{(2)} = \boldsymbol{x}_k^{(2)} - \boldsymbol{x}_j^{(2)}$$

  - New normalization:

$$\tau_j = \frac{\exp(\delta_j)}{1 + \sum_{j'} \exp(\delta_{j'})}, \forall j = 0, \ldots, n.$$

  - Product $k$ attributes: $\boldsymbol{\omega}_{j,k} = \left(\tau_k, \boldsymbol{d}_{jt,k}^{(2)}\right)$
- Demand for product $j$ is a fully exchangeable function of $\boldsymbol{\omega}_j$:

$$\sigma\left(\boldsymbol{w}_j, \boldsymbol{w}_{-j}\right) = \mathcal{D}(\boldsymbol{\omega}_j)$$

where $\boldsymbol{\omega}_j = \{\omega_{j,0}, \ldots, \omega_{j,j-1}, \omega_{j,j+1}, \ldots, \omega_{j,n}\}$.

# Main Theory Result

- Define the *exogenous* state of the market facing product $j$:

$$\begin{aligned} \boldsymbol{d}_{j,k} &= \boldsymbol{x}_k - \boldsymbol{x}_j \\ \boldsymbol{d}_j &= (\boldsymbol{d}_{j,0}, \ldots, \boldsymbol{d}_{j,j-1}, \boldsymbol{d}_{j,j+1}, \ldots, \boldsymbol{d}_{j,n}) \end{aligned}$$

**Theorem**

*If the distribution of $\{\xi_j\}_{j=1,\ldots,n}$ is exchangeable (conditional on $x_{jt}$), then the reduced form becomes*

$$E\left[\sigma_j^{-1}\left(\boldsymbol{s}, \boldsymbol{x}^{(2)}; \boldsymbol{\Sigma}^0\right)|\boldsymbol{x}\right] = g\left(\boldsymbol{d}_j\right)$$

*where $g$ is a* **symmetric** *function of the state vector.*

- **Implication:** $g$ is a *vector symmetric function* (see Briand 2009)

# Why is it useful?

1. **Curse of dimensionality:** The number of basis functions necessary to approximate the reduced-form is *independent* of the number of products (Pakes (1994), Altonji and Matzkin (2005)).
2. **Example:** Single dimension $d_{jt} = \{x_{1t} - x_{jt}, x_{2t} - x_{jt}, \ldots, x_{J_t,t} - x_{jt}\}$
   - First-order approximation of $g(d)$:

$$g(d_{jt}) \approx \sum_{j'} \gamma^1_{j'} d_{jt,j'} = \gamma^1 \left( \sum_{j'} d_{jt,j'} \right)$$

   - Second-order approximation of $g(d)$:

$$\begin{aligned} g(d_{jt}) &\approx \sum_{j'} \gamma^1_{j'} d_{jt,j'} + \sum_{j'} \gamma^2_{j'} (d_{jt,j'})^2 + \gamma^3 \left( \sum_{j'} d_{jt,j'} \right)^2 \\ &= \gamma^1 \left( \sum_{j'} d_{jt,j'} \right) + \gamma^2 \left( \sum_{j'} (d_{jt,j'})^2 \right) + \gamma^3 \left( \sum_{j'} d_{jt,j'} \right)^2 \end{aligned}$$

# Closing the loop: What is a relevant IV?

- Let $A_j(\boldsymbol{x}_t)$ be an $L$ vector of basis functions summarizing the empirical distribution of characteristic differences: $\{\boldsymbol{d}_{jt,k}\}_{k=0,\ldots,J_t}$.
- **Differentiation IV:** These functions are moments describing the relative isolation of each product in characteristic space.
- **Donald, Imbens, and Newey (2003):** Using basis functions directly as IVs, is asymptotically equivalent to approximating the optimal IV.
  - Recommended practice is to use low-order basis functions (Donald, Imbens, and Newey 2008).

# Practical Suggestions: Polynomial Basis

- **Note:** In general, the first-order basis is weak because it does not vary across products within markets (i.e. sum).
  - In practice you might still want to include the first-order basis terms when you have a lot of entry/exit of products (perhaps interacting with other distance measures).
  - Could be useful also to identify random coefficient on the intercept (rough intuition).
- Single dimension measures of differentiation

$$\text{Quadratic: } A_j(x_t) = \sum_{j'} \left( d^k_{jt,j'} \right)^2$$

Note: $\sqrt{z_{jt,k}}$ is the Euclidian distance between product $j$ and its rivals in market $t$ along dimension $k$.

- Adding interaction terms:

$$\text{Covariance: } A_j(x_t) = \sum_{j'} d^k_{jt,j'} \times d^l_{jt,j'}$$

# Practical Suggestions: Histogram Basis

- **Note:** This approach is advisable only in very large samples, and when the goal is to estimate a very flexible distribution of RCs.
- Single dimension measure of differentiation = Number of rivals in discrete bins

$$A_j(x_t) = \left\{ \sum_{j'} 1\left( d^k_{jt,j'} < \kappa_l \right) \right\}_{l=1,\ldots,L}$$

- Multi-dimension measure of differentiation:

$$A_j(x_t) = \left\{ \sum_{j'} 1\left( d^k_{jt,j'} < \kappa_l \right) 1\left( d^{k'}_{jt,j'} < \kappa_{l'} \right) \right\}_{l=1,\ldots,L,l'=1,\ldots,L}$$

# Practical Suggestions: Local Basis

- **Note:** In most parametric models, the inverse demand is function of characteristics of **close-by** rivals. Therefore, in the previous histogram basis, we should be focussing on "local" rivals.
- Single dimension measure of differentiation = Number of nearby rivals along each dimension

$$A_j(x_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right), \text{ e.g. } \kappa_k = sd(x_{jt,k})$$

- Multi-dimension measure of differentiation:

$$A_j(x_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times d_{jt,l}, \text{ e.g. } \kappa_k = sd(x_{jt,k})$$

- When $x_{jt,k}$ is discrete, this basis function boils down to the familiar Nested-logit IVs.
  - Number of competitors and characteristics of rivals within segment. See Bresnahan, Stern, and Trajtenberg (1997).

# Practical Suggestions: Demographics

- In many settings, product characteristics are fixed across markets, but the distribution of consumer types vary (e.g. Nevo 2001).
- To fix ideas, focus on a single non-linear characteristics $x_j^{(2)}$
- Consumer valuation for $x_j^{(2)}$ is

$$\beta_{it} = z_{it}\pi + \nu_i$$

  where $\nu_i \sim N(0, \sigma_x^2)$.
- **Assumption:** The distribution of demographics across markets is known, and can be decomposed as follows: $z_{it} = \mu_t + \text{sd}_t e_{it}$, where $e_{it} \sim F(\cdot)$ and $F(\cdot)$ is common across markets.
  - Example: BLP95 assume that the income distribution is log-normal with market-specific mean/variance.

# Practical Suggestions: Demographics

- Demand function:

$$\sigma_{jt}(\delta_t, x^{(2)}|\pi, \sigma_x) =$$

$$= \int\int \frac{\exp\left(\delta_{jt} + z_{it}\pi x_j^{(2)} + \nu_i x_j^{(2)}\right)}{1 + \sum_{j'} \exp\left(\delta_{j't} + z_{it}\pi x_{j'}^{(2)} + \nu_i x_{j'}^{(2)}\right)} dF_t(z_{it})\phi(\nu_i; \Sigma)$$

$$= \int\int \frac{\exp\left(\delta_{jt} + \pi e_{it}\sigma_t x_j^{(2)} + \pi\mu_t x^{(2)} + \nu_i x_j^{(2)}\right)}{1 + \sum_{j'} \exp\left(\delta_{j't} + \pi e_{it}\sigma_t x_{j'}^{(2)} + \pi\mu_t x_{j'}^{(2)} + \nu_i x_{j'}^{(2)}\right)} dF(e_{it})\phi(\nu_i; \Sigma)$$

$$= \sigma_j(\delta_j, x^{(2)}, \underbrace{\sigma_t x^{(2)}, \mu_t x^{(2)}}_{\text{new state variables}} |\pi, \sigma_x)$$

- **Implication:** Demand is a symmetric function of characteristic differences and demographic moments: $\mathcal{D}(\omega_t, d^{(2)}, \sigma_t d^{(2)}, \mu_t d^{(2)}|\theta)$.
- The previous result therefore applies to the reduced-form of this transformed model:

$$E\left[\sigma_{jt}^{-1}(s_t, x^{(2)}|\pi, \sigma_x)|x_t, \mu_t, \sigma_t\right] = g(d_t, \mu_t d^{(2)}, \sigma_t d^{(2)})$$

# Practical Suggestions: Demographics

- Differentiation IVs with demographics:

$$A_t(x_t, \mu_t, \sigma_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times \mu_t$$

$$A_t(x_t, \mu_t, \sigma_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times \sigma_t$$

$$A_t(x_t, \mu_t, \sigma_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times \sigma_t \times d^l_{jt,j'}$$

- When the distribution of demographics can be "standardized" across markets, this characterization is exact.
- More generally, Differentiation IVs should be interacted with rich moments of the distribution of consumer characteristics.
- **Example:** Miravete, Seim, and Thurk (2017)
  - Combine nested-logit 'type' instruments, with moments of the distribution of demographics across stores.

# Experiment 1: Independent Random Coefficients

- Random coefficient model:

$$u_{ijt} = \delta_{jt} + \sum_{k=1}^{K} v_{ik} x_{jt,k}^{(2)} + \varepsilon_{ijt}, \quad v_i \sim N(0, \sigma_x^2 \boldsymbol{I}).$$

- Data:
  - Panel structure: 100 markets $\times$ 15 products
  - Characteristics: $(\xi_{jt}, \boldsymbol{x}_{jt}) \sim N(0, \mathbf{I})$.
  - Dimension: $|\boldsymbol{x}_{jt}| = K + 1$
  - Monte-Carlo replications $=$ 1,000
- Differentiation IVs $(K + 1)$:
  - **Quadratic:** $A_j(\boldsymbol{x}_t) = \sum_{j'=1}^{J_t} \left(d_{jt,j'}^k\right)^2, \forall k = 1, \ldots, K$

# Simulation Results: Quadratic Differentiation IVs

| | $K_2 = 1$ | | $K_2 = 2$ | | $K_3 = 3$ | | $K_3 = 4$ | |
|---|---|---|---|---|---|---|---|---|
| | bias | rmse | bias | rmse | bias | rmse | bias | rmse |
| $\log \sigma_1$ | 0.00 | 0.03 | -0.00 | 0.03 | -0.00 | 0.03 | -0.00 | 0.04 |
| $\log \sigma_2$ | | | -0.00 | 0.03 | 0.00 | 0.03 | -0.00 | 0.04 |
| $\log \sigma_3$ | | | | | -0.00 | 0.03 | -0.00 | 0.03 |
| $\log \sigma_4$ | | | | | | | -0.00 | 0.04 |
| | | | | | | | | |
| $\sigma_1$ | 0.00 | 0.12 | 0.00 | 0.13 | -0.00 | 0.13 | -0.00 | 0.14 |
| $\sigma_2$ | | | -0.00 | 0.13 | 0.00 | 0.13 | -0.00 | 0.14 |
| $\sigma_3$ | | | | | 0.00 | 0.13 | -0.00 | 0.14 |
| $\sigma_4$ | | | | | | | -0.00 | 0.15 |
| | | | | | | | | |
| 1(Local) | 0.00 | | 0.00 | | 0.00 | | 0.00 | |
| Rank-test | 1202.10 | | 564.03 | | 330.40 | | 206.42 | |
| pv | 0.00 | | 0.00 | | 0.00 | | 0.00 | |
| IIA-test | 359.41 | | 363.22 | | 321.73 | | 276.13 | |
| pv | 0.00 | | 0.00 | | 0.00 | | 0.00 | |

# Experiment 2: Correlated Random Coefficients

- Consumer heterogeneity:

$$\beta_i^{(2)} \sim N(\beta^{(2)}, \boldsymbol{\Sigma})$$

- 4 dimensions $\Rightarrow$ 10 non-linear parameters (choleski)
- Panel structure:

$$100 \text{ markets} \times 50 \text{ products}$$

- **Differentiation IVs:** Second-order polynomials (with interactions):

$$A_j(x_t) = \sum_{j'=1}^{J_t} \left( d_{jt,j'}^k \times d_{jt,j'}^l \right)$$

for all characteristics $k <= l$.

# Simulation Results: Correlated Random-Coefficients

| | $\Sigma_{\cdot,1}$ | $\Sigma_{\cdot,2}$ | $\Sigma_{\cdot,3}$ | $\Sigma_{\cdot,4}$ |
|---|---|---|---|---|
| $\Sigma_{1,\cdot}$ | 0.003 | 0.003 | -0.003 | 0.010 |
| $\Sigma_{2,\cdot}$ | 0.003 | 0.000 | 0.004 | -0.000 |
| $\Sigma_{3,\cdot}$ | -0.003 | 0.004 | -0.009 | 0.006 |
| $\Sigma_{4,\cdot}$ | 0.010 | -0.000 | 0.006 | 0.010 |
| $\Sigma_{1,\cdot}$ | 0.228 | 0.132 | 0.156 | 0.156 |
| $\Sigma_{2,\cdot}$ | 0.132 | 0.232 | 0.145 | 0.143 |
| $\Sigma_{3,\cdot}$ | 0.156 | 0.145 | 0.217 | 0.154 |
| $\Sigma_{4,\cdot}$ | 0.156 | 0.143 | 0.154 | 0.217 |
| IIA test (F) | 157.637 | | | |
| Cragg-Donald stat. | 474.053 | | | |
| Nb endo. | 10.000 | | | |
| Nb IVs | 15.000 | | | |

# How to account for endogenous prices?

- "Quality-ladder" example:

$$u_{ijt} = \delta_{jt} - \alpha_i p_{jt} + \varepsilon_{ijt}$$

where $\alpha_i = \sigma_p y_i^{-1}$, and $\log(y_i) \sim \mathrm{N}(\mu_y, \sigma_y)$ (known).

- Excluded price instruments: $\boldsymbol{w}_t = \{w_{jt}\}_{j=1,\dots,J_t}$

- Reduced-form:

$$E\left[\sigma_j^{-1}\left(\boldsymbol{s}_t, \boldsymbol{x}_t, \boldsymbol{p}_t | \sigma_p^0\right) | \boldsymbol{x}_t, \boldsymbol{w}_t\right] \neq g(\boldsymbol{d}_j^x, \boldsymbol{d}_j^p),$$

where the inequality is due to the simultaneity of prices and $\xi_{jt}$ (BLP, 1995).

# How to incorporate endogenous prices?

- **Heuristic solution:** Distribute the expectation for price *inside* of the inverse-demand (BLP, 1999):

$$
\begin{aligned}
E\left[\sigma_j^{-1}(\boldsymbol{s}_t, \boldsymbol{p}_t, \boldsymbol{x}_t^{(2)}; \Sigma) | \boldsymbol{x}_t, \boldsymbol{w}_t\right] &\approx E\left[\sigma_j^{-1}(\boldsymbol{s}_t, \hat{\boldsymbol{p}}_t, \boldsymbol{x}_t^{(2)}; \Sigma) | \boldsymbol{x}_t, \hat{\boldsymbol{p}}_t\right] \\
&= g(\boldsymbol{d}_{jt}^{x}, \boldsymbol{d}_{jt}^{\hat{p}})
\end{aligned}
$$

where $\hat{d}_{jt,k}^{p} = E(p_{kt} | \boldsymbol{w}_{kt}) - E(p_{jt} | \boldsymbol{w}_{jt})$.

# Experiment 3: Differentiation IVs with Endogenous Prices

## Example with cost shifter

1. Exogenous price index (OLS):

$$\hat{p}_{jt} = \hat{\pi}_0 + \hat{\pi}_1 x_{jt} + \hat{\pi}_2 \omega_{jt}$$

2. Differentiation IV: Quadratic

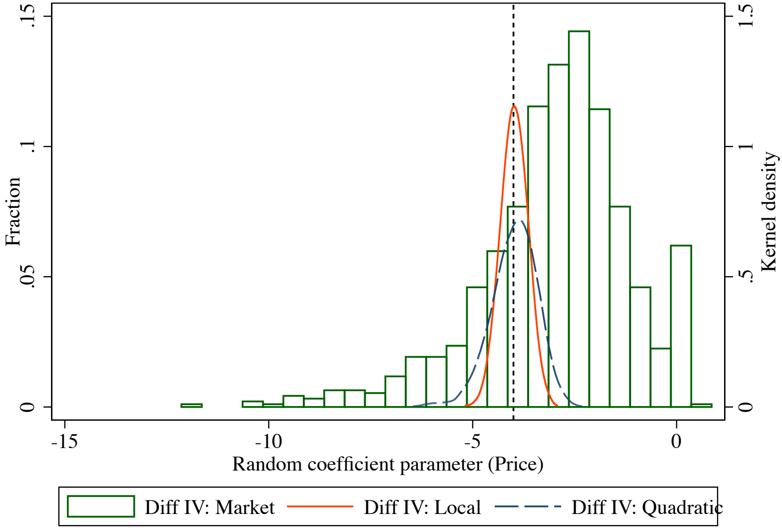
$$\sum_{j'} \left( d^{\hat{p}}_{jt,j'} \right)^2 \text{ and } \sum_{j'} \left( d^{\hat{p}}_{jt,j'} \right)^2 \cdot \boldsymbol{d}_{jt,j'}$$

   where $\boldsymbol{d}_{jt,j'} = (d^{x}_{jt,j'}, d^{\hat{p}}_{jt,j'})$.

3. Differentiation IV: Local

$$\sum_{j'} \left( |d^{\hat{p}}_{jt,j'}| < \text{sd}(\hat{p}_{jt}) \right) \text{ and } \sum_{j'} \left( |d^{\hat{p}}_{jt,j'}| < \text{sd}(\hat{p}_{jt}) \right) \cdot \boldsymbol{d}_{jt,j'}$$

# Distribution of $\hat{\sigma}_p$ with weak and strong IVs




Dash vertical line = True parameter value

# GMM estimates with endogenous prices

| | True | Diff. IV = Local | | | Diff. IV = Quadratic | | | Diff. IV = Sum | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | bias | se | rmse | bias | se | rmse | bias | se | rmse |
| $\sigma_p$ | -4.00 | 0.02 | 0.27 | 0.28 | 0.02 | 0.53 | 0.55 | 1.03 | 158.25 | 2.10 |
| $\beta_0$ | 50.00 | -0.26 | 3.92 | 3.92 | -0.28 | 7.36 | 7.45 | -9.82 | 26.41 | 20.65 |
| $\beta_x$ | 2.00 | -0.02 | 0.46 | 0.45 | -0.02 | 0.47 | 0.47 | 0.34 | 1.11 | 0.83 |
| $\beta_p$ | -0.20 | 0.01 | 0.37 | 0.37 | 0.01 | 0.31 | 0.32 | -0.67 | 201.29 | 1.38 |

# GMM estimates with endogenous prices

| | Diff. IV = Local | Quadratic | Diff. IV = Sum |
|---|---|---|---|
| Frequency conv. | 1 | 1 | 0.94 |
| IIA-test | 109.48 | 53.90 | 1.88 |
| p-value | 0 | 0 | 0.34 |
| 1st-stage F-test: Price | 191.80 | 442.10 | 138.94 |
| 1st-stage F-test: Jacobian | 214.60 | 58.40 | 27.85 |
| Cond. 1st-stage F-test: Price | 252.23 | 479.96 | 7.92 |
| Cond. 1st-stage F-test: Jacobian | 280.31 | 82.44 | 6.19 |
| Cragg-Donald statistics | 170.19 | 54.45 | 4.09 |
| Stock-Yogo size CV (10%) | 16.87 | 13.43 | 13.43 |
| Nb. endogenous variables | 2 | 2 | 2 |
| Nb. IVs | 4 | 3 | 3 |

**Note:** The *Conditional 1st-stage F-test* statistic is the Weak IV test proposed by Angrist and Pischke for multiple endogenous variables.

# Experiment 4: Natural Experiments

- **Hotelling example:** Exogenous entry of a new product ($x' = 5$)

$$u_{ijmt} = \delta_{jmt} - \lambda|\nu_i - x_{jmt}| + \epsilon_{ijmt}$$

- Three-way panel: product $j$, market $m$, and time ($t = 0, 1$).
- Treatment variable:

$$D_{jm} = 1\left(|x_{jm} - 5| < \text{Cutoff}\right)$$

- Reduced-form: Difference-in-difference regression

$$\sigma_j^{-1}(s_t, x_t, p_t|\theta^0) = \mu_{jm} + \tau_t + \gamma D_{jm} \times 1(t = 1) + \xi_{jmt}$$

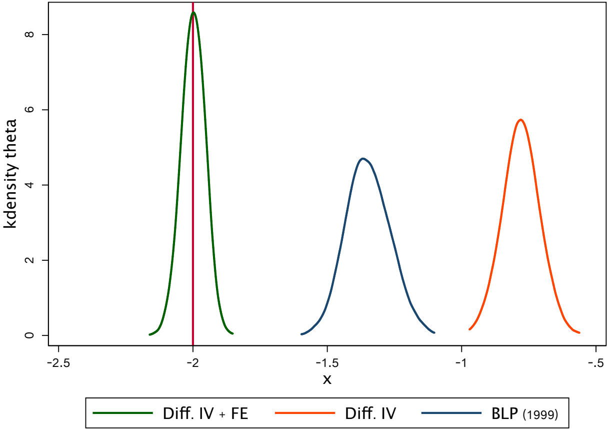
- **GMM:** DiD IVs
  - Linear characteristics: $x_{jmt}^{(1)}$ = Market/Product FE + After Dummy
  - Differentiation IV: $z_{jmt} = D_{jm} \times 1(t = 1)$
  - $\hat{\theta}^{gmm}$ is identified from the DiD variation in $z_{jmt}$.

# Natural Experiment: Hotelling Example

DGP: $\delta_{jmt} = \bar{\xi}_{jm} + \Delta\xi_{jmt}$, where $E(\bar{\xi}_{jm}|\boldsymbol{x}_m) \neq 0$

- "Diff-in-Diff" specification:

$$\boldsymbol{z}_{jmt} = \{\text{Product Dummy}_{jm}, 1(t=1), 1(|x_{jm} - 5| < 1)1(t=1)\}$$



# Review of Nonlinear GMM

- The GMM problem is defined as:

$$\min_{\beta,\Sigma} \quad = \quad \rho(\theta)' Z \hat{W}^{-1} Z' \rho(\theta)$$
$$\text{s.t.} \qquad \rho_j(s_t, x_t|\theta) = \sigma_j^{-1}(s_t, x_t^{(2)}|\Sigma) - x_{jt}\beta$$

- The FOC of the problem is (or moment conditions):

$$\frac{\partial \rho(\theta)'}{\partial \theta} Z \hat{W}^{-1} Z' \rho(\theta) = 0$$

where $\frac{\partial \rho(\theta)}{\partial \theta}$ is a $n \times m$ matrix containing the **Jacobian** of the residual function.

- In our context:

$$\frac{\partial \rho_j(s_t, x_t|\theta)}{\partial \theta} = \begin{cases} -x_{jt,k} & \text{If } \theta_k = \beta_k \\ \frac{\partial \sigma_j^{-1}(s_t, x_t \theta_k)}{\partial \Sigma_k} & \text{Else.} \end{cases}$$

The derivative of the inverse demand can be computed using the implicit function theorem (see Nevo (2001)).

# Review of Nonlinear GMM

- Let $J_{jt}(\theta) = \frac{\partial \rho_j(s_t, x_t|\theta)}{\partial \theta}$ denotes the matrix of Jacobian.
- Notice that the moment conditions imposed by GMM correspond to the moment conditions associated with a **linear approximation** of the model.
- The moment conditions at $\theta^{gmm}$ is:

$$J(\theta)' Z \hat{W}^{-1} Z' \rho(\theta) = 0$$

- This is the moment condition of the following linear IV regression:

$$\begin{aligned} \rho_{jt}(\theta) &= J_{jt}(\theta) b + \text{Error} \\ &= \sum_k (\theta_k - \theta_k^0) \frac{\partial \rho_j(s_t, x_t|\theta)}{\partial \theta} + \text{Error} \end{aligned}$$

  where $b = 0$.
- Therefore, $b^{gmm} = E(\theta^{gmm} - \theta^0) = 0$ (i.e. GMM is consistent).

# Gauss-Newon Regression

- The linear representation can be used to construct a Gauss-Newton algorithm to estimate $\{\hat{\theta}\}$.

$$\rho_{jt}(\theta) \quad = \quad J_{jt}(\theta)b + \text{Error}$$

- Iteration $k \geq 1$:
  1. Invert demand: $\sigma_{jt}^{-1}(s_t, x_t|\theta^{k-1})$ and $J_{jt}(s_t, x_t|\theta^{k-1})$
  2. Estimate $\{\hat{b}^k\}$ by linear GMM.
  3. If $|\hat{b}^k| < \varepsilon$ stop. Else, update $\theta^k = \theta^{k-1} + \hat{b}^k$, and repeat steps (1)-(3).
- With strong instruments, this procedure typically requires less than 5 iterations.
  - Weak IVs lead to severe numerical problems (e.g. Knittle and Metaxoglou (2014), Dube et al. (2012))
  - Why? The central-limit theorem does no hold, and the quadratic approximation is bad.

# Using the Gauss-Newton Regression for Inference

- The Gauss-Newton regression is also useful to conduct inference
- Let $\hat{\theta}$ and $\hat{W}$ denote GMM estimate and weighting matrix (estimated using Julia or any other non-linear optimization package)
- Since non-linear GMM is equivalent to linear IV at the solution, we can conduct inference on $\hat{\theta}$ using the GNR:

$$\rho_{jt}(\hat{\theta}) = J_{jt}(\hat{\theta})\hat{b} + \text{Error}$$

- **Note:** $\hat{b}$ must be **zero** if $\hat{\theta}$ is the GMM solution (i.e. you must be using the same weighing matrix in Julia or Matlab as in STATA or R to run this regression)
- But, variance-covariance matrix of $\hat{\theta}$ is the same as the variance-covariance matrix of $\hat{\theta}$. Therefore, you can use standard statistical routines to calculate standard errors and conduct hypothesis tests on $\theta$ (e.g. cluster standard errors, equality of parameters, etc.).

# Using the Gauss-Newton Regression for to evaluate the Strength of IVs

- This insight is particularly useful to evaluate the relevance of the instruments
- Some weak identification tests are non trivial to code, and are standard in STATA and R
- **Examples:** IVREG2 in STATA reports the Cragg-Donald and the Sanderson-Windmeijer (SW) first-stage tests.
- **Recommeded procedure:**
  - Use Julia's non-linear optimization packages to solve the GMM problem
  - Compute the Jacobian of the inverse demand
  - Expor the Jacobian and weighting matrix to STATA or R (or even better load R in Julia...)
  - Perform inference and weak IV tests using the Gauss-Newton regression

# Ex-Ante Weak IV test: IIA Hypothesis

- **Warning:** The weak IV test based on the Jacobian function at $\hat{\theta}$ are not consistent when the instruments are too weak. You should therefore not put too much weight on the p-values, and combine your analysis with the IIA test.
- A **strong** instrument for $\Sigma$ is able to reject the wrong model (Stock and Wright, 2000)
- Under $H_0 : \Sigma = 0$, the inverse demand equation is independent of $x_{-j}$:

$$\sigma_{jt}^{-1}(s_t, x_t; \Sigma = 0) = \ln s_{jt}/s_{0t} = x_{jt}\beta + z_{jt}\gamma + \xi_{jt}$$

- Standard test statistics for $H_0 : \gamma = 0$, can be used to test null hypothesis of IIA preferences
- With endogenous prices, this test is equivalent to the J-test at $\Sigma = 0$.
- In practice, you should report both this *ex-ante* weak IV test, and the *ex-post* specification tests.

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