Measuring Substitution Patterns in Differentiated Product Industries

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Motivation

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- **The Beauty of BLP:** Flexible estimation of substitution patterns with *many* products, aggregate data, and unobserved attributes.
  - Workhorse model to study demand for differentiated products in IO
  - Increasingly used to analyse *sorting* problems in urban, education, insurance, etc.

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  - Workhorse model to study demand for differentiated products in IO
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- Achieving this flexibility can be difficult in practice...
  - **Precision:** Often rely on external restrictions (e.g. supply, survey, etc.)
  - **Numerical:** Multiple solutions and/or poor convergence properties

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- **Our paper** argue that many *empiricists'* problems are caused by **weak** IVs
- Show how to construct **strong** IVs using a new representation of the reduced-form of the model.

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  - **Differentiation IV:** *Nested-Logit* (e.g. Berry (1994), Verboven (1996), Bresnahan et al. (1997)), and *Spatial Differentiation* (e.g. Pinkse and Slade (2001), Davis (2006), Thomadsen (2005), Manuszak (2012), Houde (2012))

Baseline Model: Exogenous Characteristics

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$$\sigma_j\left(\delta_t, x_t^{(2)}; \lambda\right) = \int \frac{\exp\left(\delta_{jt} + \nu_i^T x_{jt}^{(2)}\right)}{1 + \sum_{j'=1}^{J_t} \exp\left(\delta_{j't} + \nu_i^T x_{j't}^{(2)}\right)} dF(\nu_i|\lambda)$$

where $\delta_{jt} = \beta_0 + x_{jt}^{(1)}\beta_1 + x_{jt}^{(2)}\beta_2 + \xi_{jt}$.

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where $\delta_{jt} = \beta_0 + x_{jt}^{(1)}\beta_1 + x_{jt}^{(2)}\beta_2 + \xi_{jt}$.

- The **residual** of the model is obtained from the inverse-demand function:

$$\rho_j(s_t, x_t; \theta) = \sigma_j^{-1}\left(s_t, x_t^{(2)}; \lambda\right) - x_{jt}\beta, \quad \text{where } \theta = (\beta, \lambda).$$

Identifying Assumption

- **Assumption:** The unobserved attribute of each product is independent of the **menu**, x_t, of characteristics available in market t,

$$E[\xi_{jt}|x_t] = 0 \quad \text{(CMR)}.$$

- In practice, the model is estimated using a finite number (L) of unconditional moment restrictions, $A_j(x_t)$:

$$\begin{aligned} & E\left[\rho_j(s_t, x_t; \theta^0) \cdot A_j(x_t)\right] = 0 \\ \leftrightarrow \quad & E\left[\left(\sigma_j^{-1}\left(s_t, x_t^{(2)}; \lambda^0\right) - x_{jt}\beta\right) \cdot A_j(x_t)\right] = 0. \end{aligned}$$

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- **Our question:** How to construct relevant instruments to identify λ?
  - **Stock & Wright (2000)**: $A_j(x_t)$ is weak if the moment conditions are *almost* satisfied away from the true parameters.

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$$H_0 : E\left[\rho_j(s_t, x_t|\beta, \lambda = 0) \cdot z_{jt}\right] = 0$$

$$\leftrightarrow \qquad \ln s_{jt}/s_{0t} = x_{jt}\beta + \gamma z_{jt} + \xi_{jt}$$

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  2. Local identification: Cragg-Donald rank test

$$\text{rank}\left(E\left[\partial\rho_j\left(s_t, x_t; \theta\right)/\partial\theta^T \cdot z_{jt}\right]\right) = m$$

$$\leftrightarrow J_{jt,k}(\theta) = z_t\pi^k + u_{jt,k}$$

where π^k are the "reduced-form" parameters of the model. This test can be implemented in STATA (ivreg2 or ranktest).

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- Monte-Carlo design:
  - Sample: $T = 100$ and $J = 15$
  - Random utility with (independent) normal random-coefficients (K_2)
  - DGP: $(x_{jt,k}, \xi_{jt}) \sim N(0, I)$ [homoscedasticity]

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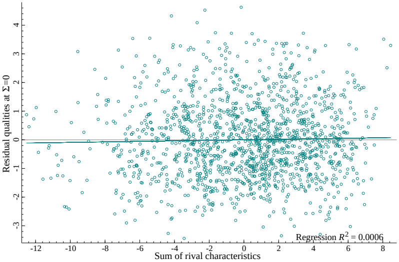
(A) IV: Sum of rivals' characteristics



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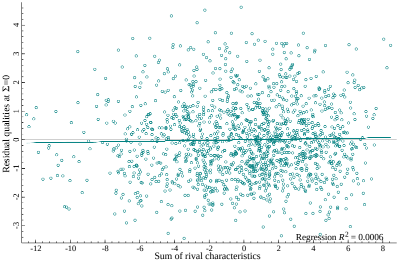
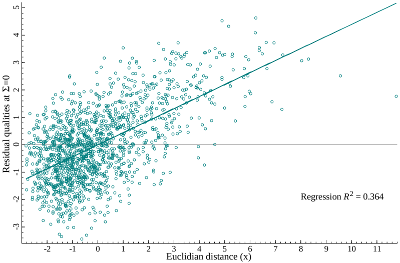


(B) IV: Euclidean distance in x



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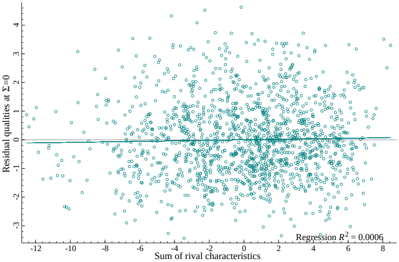

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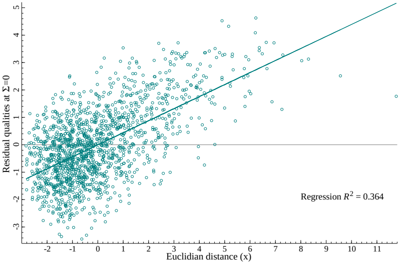
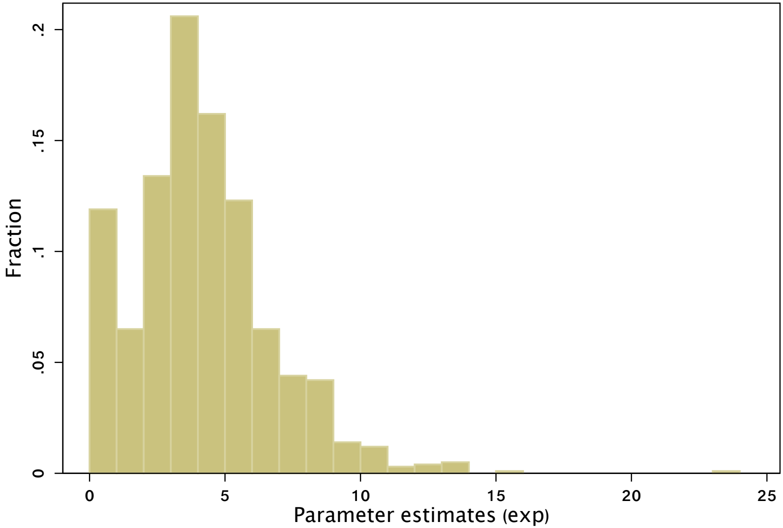


- **Takeaway:** Independence of ξ_{jt} and the *distance of rival characteristics* rules out the IIA hypothesis, but **not** the *sum of rival characteristics*.

Distribution of $\hat{\sigma}_2$ with weak IVs



Shapiro-Wilk test for normality: 15.71 (0). Width = 1.

GMM Estimates with Weak IVs

	$K_2 = 1$		$K_2 = 2$		$K_2 = 3$		$K_2 = 4$	
	bias	rmse	bias	rmse	bias	rmse	bias	rmse
$\log \sigma_1$	-11.29	95.93	-5.43	74.95	-1.15	5.50	-8.40	229.67
$\log \sigma_2$			-4.69	58.31	-1.36	6.26	-1.10	6.17
$\log \sigma_3$					-1.41	9.20	-4.66	112.64
$\log \sigma_4$							-0.93	4.02
σ_1	0.14	2.64	-0.01	2.49	-0.03	2.19	0.22	2.35
σ_2			0.12	2.42	-0.01	2.27	0.10	2.30
σ_3					0.18	2.38	0.11	2.38
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1(Local-min)	0.19		0.51		0.59		0.66	
Range(J)	0.74		1.15		1.64		1.51	
Range(pv)	0.17		0.19		0.21		0.21	
Range($\log \sigma$)	11.74		6.64		6.58		4.86	
Rank-test	1.26		0.46		0.26		0.18	
p-value	0.62		0.81		0.89		0.92	
IIA-test	1.33		1.30		1.49		1.94	
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Identification problem

- **Simultaneous equation:** Reduced-form *vs* structural equation

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- **Example:** Quasi-linear utility with exogenous prices

$$u_{ijt} = x_{jt}b_i - p_{jt} + \xi_{jt} + \epsilon_{ijt}; b_i = \beta + \lambda\eta_i$$

$$\rightarrow p_{jt} = x_{jt}\beta + \sigma_j^{-1}(s_t, x^{(2)}; \lambda) + \xi_{jt} = \text{Non-linear IV regression}$$

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- **Insight from Berry & Haile:** The presence of a *special regressor* $x_{jt}^{(1)}$ implies that $x_{-j,t}^{(1)}$ can be used as *excluded instruments* for the endogenous shares.

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- **Definition:** An "efficient" instrument is a (basis) function $A^L(x_t)$ of dimension L, that can approximate the reduced-form of the model arbitrarily well:

$$E\left[\sigma_j^{-1}\left(s_t, x_t^{(2)}; \lambda^0\right)|x_t\right] - A_j^L(x_t)\hat{\gamma}_L \rightarrow 0, \text{ as } L \text{ and } n \text{ get large.}$$

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  - Where $\hat{\gamma}_L$ are OLS coefficients obtained by projecting σ^{-1} onto $A_j^L(x_t)$.
- The same basis functions can be used to construct Chamberlain (1987)'s optimal instruments. See Newey (1990).

Curse of Dimensionality Problem

- **Curse of Dimensionality:** The reduced-form is a *product-specific* function of the entire menu of product characteristics.
  - As $J \uparrow$, both the number of arguments **and** the number of functions to approximate increase.
- Without further restrictions, we cannot directly use the insights of BH to construct relevant IVs

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- Market-structure facing product j (dropping t):

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  $$\sigma (\boldsymbol{w}_j, \boldsymbol{w}_{-j}) = \sigma \left(\boldsymbol{w}_j, \boldsymbol{w}_{\rho(-j)} \right) \forall \rho$$
  - Translation invariant: for any $\boldsymbol{c} \in \mathbb{R}^K$
  $$\sigma (\boldsymbol{w}_j + (0, \boldsymbol{c}), \boldsymbol{w}_{-j} + (0, \boldsymbol{c})) = \sigma (\boldsymbol{w}_j, \boldsymbol{w}_{-j})$$

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  - New normalization:

$$\tau_j = \frac{\exp(\delta_j)}{1 + \sum_{j'} \exp(\delta_{j'})}, \forall j = 0, \ldots, n.$$

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- Demand for product j is a fully exchangeable function of $\boldsymbol{\omega}_j$:

$$\sigma\left(\boldsymbol{w}_j, \boldsymbol{w}_{-j}\right) = \mathcal{D}(\boldsymbol{\omega}_j)$$

where $\boldsymbol{\omega}_j = \{\omega_{j,0}, \ldots, \omega_{j,j-1}, \omega_{j,j+1}, \ldots, \omega_{j,n}\}$.

Main Theory Result

- Define the *exogenous* state of the market facing product j:

$$
\begin{aligned}
\boldsymbol{d}_{j,k} &= \boldsymbol{x}_k - \boldsymbol{x}_j \\
\boldsymbol{d}_j &= (\boldsymbol{d}_{j,0}, \ldots, \boldsymbol{d}_{j,j-1}, \boldsymbol{d}_{j,j+1}, \ldots, \boldsymbol{d}_{j,n})
\end{aligned}
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Theorem

If the distribution of $\{\xi_j\}_{j=1,\ldots,n}$ is exchangeable (conditional on x_{jt}), then the reduced form becomes

$$E\left[\sigma_j^{-1}\left(\boldsymbol{s}, \boldsymbol{x}^{(2)}; \boldsymbol{\lambda}^0\right)|\boldsymbol{x}\right] = g\left(\boldsymbol{d}_j\right)$$

where g is a **symmetric** *function of the state vector.*

Main Theory Result

- Define the *exogenous* state of the market facing product j:

$$\begin{aligned} \boldsymbol{d}_{j,k} &= \boldsymbol{x}_k - \boldsymbol{x}_j \\ \boldsymbol{d}_j &= (\boldsymbol{d}_{j,0}, \ldots, \boldsymbol{d}_{j,j-1}, \boldsymbol{d}_{j,j+1}, \ldots, \boldsymbol{d}_{j,n}) \end{aligned}$$

Theorem

If the distribution of $\{\xi_j\}_{j=1,\ldots,n}$ is exchangeable (conditional on x_{jt}), then the reduced form becomes

$$E\left[\sigma_j^{-1}\left(\boldsymbol{s}, \boldsymbol{x}^{(2)}; \boldsymbol{\lambda}^0\right)|\boldsymbol{x}\right] = g\left(\boldsymbol{d}_j\right)$$

where g is a **symmetric** *function of the state vector.*

- **Implication:** g is a *vector symmetric function* (see Briand 2009)

Why is it useful?

1. **Curse of dimensionality:** The number of basis functions necessary to approximate the reduced-form is *independent* of the number of products and markets (Pakes (1994), Altonji and Matzkin (2005)).

Why is it useful?

1. **Curse of dimensionality:** The number of basis functions necessary to approximate the reduced-form is *independent* of the number of products and markets (Pakes (1994), Altonji and Matzkin (2005)).
2. **Example:** Single dimension $d_{jt} = \{x_{1t} - x_{jt}, x_{2t} - x_{jt}, \ldots, x_{J_t,t} - x_{jt}\}$
   - First-order approximation of $g(d)$:

$$g(d_{jt}) \approx \sum_{j'} \gamma^1_{j'} d_{jt,j'} = \gamma^1 \left(\sum_{j'} d_{jt,j'} \right)$$

   - Second-order approximation of $g(d)$:

$$\begin{aligned} g(d_{jt}) &\approx \sum_{j'} \gamma^1_{j'} d_{jt,j'} + \sum_{j'} \gamma^2_{j'} (d_{jt,j'})^2 + \gamma^3 \left(\sum_{j'} d_{jt,j'} \right)^2 \\ &= \gamma^1 \left(\sum_{j'} d_{jt,j'} \right) + \gamma^2 \left(\sum_{j'} (d_{jt,j'})^2 \right) + \gamma^3 \left(\sum_{j'} d_{jt,j'} \right)^2 \end{aligned}$$

Closing the loop: What is a relevant IV?

- Let $A_j\left(\boldsymbol{x}_t\right)$ be an L vector of basis functions summarizing the empirical distribution of characteristic differences: $\{\boldsymbol{d}_{jt,k}\}_{k=0,\ldots,J_t}$.

Closing the loop: What is a relevant IV?

- Let $A_j(\boldsymbol{x}_t)$ be an L vector of basis functions summarizing the empirical distribution of characteristic differences: $\{\boldsymbol{d}_{jt,k}\}_{k=0,\ldots,J_t}$.
- **Differentiation IV:** These functions are moments describing the relative isolation of each product in characteristic space.

Closing the loop: What is a relevant IV?

- Let $A_j(\boldsymbol{x}_t)$ be an L vector of basis functions summarizing the empirical distribution of characteristic differences: $\{\boldsymbol{d}_{jt,k}\}_{k=0,\ldots,J_t}$.
- **Differentiation IV:** These functions are moments describing the relative isolation of each product in characteristic space.
- **Donald, Imbens, and Newey (2003):** Using basis functions directly as IVs, is asymptotically equivalent to approximating the optimal IV.
    - Recommended practice is to use low-order basis functions (Donald, Imbens, and Newey 2008).

Suggestion 1: Polynomial Basis

- Single dimension measures of differentiation

$$\text{Quadratic: } A_j(x_t) = \sum_{j'} \left(d^k_{jt,j'} \right)^2$$

Note: $\sqrt{z_{jt,k}}$ is the Euclidian distance between product j and its rivals in market t along dimension k.

- Adding interaction terms:

$$\text{Covariance: } A_j(x_t) = \sum_{j'} d^k_{jt,j'} \times d^l_{jt,j'}$$

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- **Note:** In general, the first-order basis is weak because it does not vary across products within markets (i.e. sum of rival characteristics).

Suggestion 2: Histogram Basis

- Single dimension measure of differentiation = Number of rivals in discrete bins

$$A_j(x_t) = \left\{ \sum_{j'} 1\left(d^k_{jt,j'} < \kappa_l\right) \right\}_{l=1,...,L}$$

- Multi-dimension measure of differentiation:

$$A_j(x_t) = \left\{ \sum_{j'} 1\left(d^k_{jt,j'} < \kappa_l\right) 1\left(d^{k'}_{jt,j'} < \kappa_{l'}\right) \right\}_{l=1,...,L,l'=1,...,L}$$

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- **Note:** This approach is advisable only in very large samples (+large choice-sets), and when the goal is to estimate a flexible distribution of RCs (e.g. correlation terms)

Suggestion 3: Local Basis

- In most parametric models, the inverse demand is function of characteristics of **close-by** rivals. Therefore, the characteristics of "nearby" rivals should more relevant.

Suggestion 3: Local Basis

- In most parametric models, the inverse demand is function of characteristics of **close-by** rivals. Therefore, the characteristics of "nearby" rivals should more relevant.
- Single dimension measure of differentiation = Number of nearby rivals along each dimension

$$A_j(x_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right), \text{ e.g. } \kappa_k = sd(x_{jt,k})$$

- Multi-dimension measure of differentiation:

$$A_j(x_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times d_{jt,l}, \text{ e.g. } \kappa_k = sd(x_{jt,k})$$

- When $x_{jt,k}$ is discrete, this basis function boils down to the familiar Nested-logit IVs (e.g. Berry (1994), Bresnahan et al. (1997)).

Suggestion 4: Demographics

- In many settings, product characteristics are fixed across markets, but the distribution of consumer types vary (e.g. Nevo 2001).
- To fix ideas, focus on a single non-linear characteristics $x_j^{(2)}$
- Consumer valuation for $x_j^{(2)}$ is

$$\beta_{it} = z_{it}\pi + \nu_i$$

  where $\nu_i \sim N(0, \sigma_x^2)$.
- **Assumption:** The distribution of demographics across markets is known, and can be decomposed as follows: $z_{it} = \mu_t + \text{sd}_t e_{it}$, where $e_{it} \sim F(\cdot)$ and $F(\cdot)$ is common across markets.
  - Example: BLP95 assume that the income distribution is log-normal with market-specific mean/variance.

Suggestion 4: Demographics

- Demand function:

$$
\begin{aligned}
&\sigma_{jt}(\delta_t, x^{(2)}|\pi, \sigma_x) = \\
&= \int\int \frac{\exp\left(\delta_{jt} + z_{it}\pi x_j^{(2)} + \nu_i x_j^{(2)}\right)}{1 + \sum_{j'} \exp\left(\delta_{j't} + z_{it}\pi x_{j'}^{(2)} + \nu_i x_{j'}^{(2)}\right)} dF_t(z_{it})\phi(\nu_i; \lambda) \\
&= \int\int \frac{\exp\left(\delta_{jt} + \pi e_{it}\sigma_t x_j^{(2)} + \pi\mu_t x^{(2)} + \nu_i x_j^{(2)}\right)}{1 + \sum_{j'} \exp\left(\delta_{j't} + \pi e_{it}\sigma_t x_{j'}^{(2)} + \pi\mu_t x_{j'}^{(2)} + \nu_i x_{j'}^{(2)}\right)} dF(e_{it})\phi(\nu_i; \lambda) \\
&= \sigma_j(\delta_j, x^{(2)}, \underbrace{\sigma_t x^{(2)}, \mu_t x^{(2)}}_{\text{new characteristics}} |\pi, \sigma_x) \\
&= \mathcal{D}(\omega_t, d^{(2)}, \sigma_t d^{(2)}, \mu_t d^{(2)}|\theta)\text{: Symmetric function!}
\end{aligned}
$$

- The reduced-form of this transformed model can therefore be written:

$$
E\left[\sigma_{jt}^{-1}(s_t, x^{(2)}|\pi, \sigma_x)|x_t, \mu_t, \sigma_t\right] = g(d_t, \mu_t d^{(2)}, \sigma_t d^{(2)})
$$

Suggestion 4: Demographics

- Differentiation IVs with demographics:

$$A_t(x_t, \mu_t, \sigma_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times \mu_t$$

$$A_t(x_t, \mu_t, \sigma_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times \sigma_t$$

$$A_t(x_t, \mu_t, \sigma_t) = \sum_{j'} 1\left(|d^k_{jt,j'}| < \kappa_k\right) \times \sigma_t \times d^l_{jt,j'}$$

- When the distribution of demographics can be "standardized" across markets, this characterization is exact.
  - Differentiation IVs should be interacted with moments of the distribution of consumer characteristics to separately identify the two sources of heterogeneity.
- **Example:** Miravete, Seim, and Thurk (2017)
  - Combine nested-logit 'type' instruments, with moments of the distribution of demographics across stores.

Monte-Carlo Simulations

1. Independent random coefficients
2. Correlated random coefficients
3. Endogenous prices
4. Natural experiments
5. Optimal IV approximation: Comparison with Berry et al. (1999) and Reynaert and Verboven (2013).

Experiment 1: Independent Random Coefficients

- Random coefficient model:

$$u_{ijt} = \delta_{jt} + \sum_{k=1}^{K} v_{ik} x_{jt,k}^{(2)} + \varepsilon_{ijt}, \quad v_i \sim N(0, \sigma_x^2 \boldsymbol{I}).$$

- Data:
  - Panel structure: 100 markets $\times$ 15 products
  - Characteristics: $(\xi_{jt}, \boldsymbol{x}_{jt}) \sim N(0, \mathbf{I})$.
  - Dimension: $|\boldsymbol{x}_{jt}| = K + 1$
  - Monte-Carlo replications = 1,000

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  - Dimension: $|\boldsymbol{x}_{jt}| = K + 1$
  - Monte-Carlo replications = 1,000
- Differentiation IVs $(K + 1)$:
  - **Quadratic:** $A_j(\boldsymbol{x}_t) = \sum_{j'=1}^{J_t} \left(d_{jt,j'}^k\right)^2, \forall k = 1, \ldots, K$

Simulation Results: Quadratic Differentiation IVs

	$K_2 = 1$		$K_2 = 2$		$K_3 = 3$		$K_3 = 4$	
	bias	rmse	bias	rmse	bias	rmse	bias	rmse
$\log \sigma_1$	0.00	0.03	-0.00	0.03	-0.00	0.03	-0.00	0.04
$\log \sigma_2$			-0.00	0.03	0.00	0.03	-0.00	0.04
$\log \sigma_3$					-0.00	0.03	-0.00	0.03
$\log \sigma_4$							-0.00	0.04
σ_1	0.00	0.12	0.00	0.13	-0.00	0.13	-0.00	0.14
σ_2			-0.00	0.13	0.00	0.13	-0.00	0.14
σ_3					0.00	0.13	-0.00	0.14
σ_4							-0.00	0.15
1(Local)	0.00		0.00		0.00		0.00	
Rank-test	1202.10		564.03		330.40		206.42	
pv	0.00		0.00		0.00		0.00	
IIA-test	359.41		363.22		321.73		276.13	
pv	0.00		0.00		0.00		0.00	

Experiment 2: Correlated Random Coefficients

- Consumer heterogeneity:

$$\boldsymbol{\beta}_i^{(2)} \sim N(\boldsymbol{\beta}^{(2)}, \boldsymbol{\lambda})$$

- 4 dimensions $\Rightarrow$ 10 non-linear parameters (choleski)
- Panel structure:

$$100 \text{ markets} \times 50 \text{ products}$$

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- 4 dimensions $\Rightarrow$ 10 non-linear parameters (choleski)
- Panel structure:

100 markets $\times$ 50 products

- **Differentiation IVs:** Second-order polynomials (with interactions):

$$A_j(x_t) = \sum_{j'=1}^{J_t} \left(d_{jt,j'}^{k} \times d_{jt,j'}^{l} \right)$$

for all characteristics $k <= l$.

Simulation Results: Correlated Random-Coefficients

		$\Sigma_{\cdot,1}$	$\Sigma_{\cdot,2}$	$\Sigma_{\cdot,3}$	$\Sigma_{\cdot,4}$
Bias	$\Sigma_{1,\cdot}$	0.003	0.003	-0.003	0.010
	$\Sigma_{2,\cdot}$	0.003	0.000	0.004	-0.000
	$\Sigma_{3,\cdot}$	-0.003	0.004	-0.009	0.006
	$\Sigma_{4,\cdot}$	0.010	-0.000	0.006	0.010
RMSE	$\Sigma_{1,\cdot}$	0.228	0.132	0.156	0.156
	$\Sigma_{2,\cdot}$	0.132	0.232	0.145	0.143
	$\Sigma_{3,\cdot}$	0.156	0.145	0.217	0.154
	$\Sigma_{4,\cdot}$	0.156	0.143	0.154	0.217
IIA test (F)		157.637			
Rank test		474.053			
Nb endo.		10.000			
Nb IVs		15.000			

$\Sigma_{1,\cdot}$	4			
$\Sigma_{2,\cdot}$	-2	4		
$\Sigma_{3,\cdot}$	2	-2	4	
$\Sigma_{4,\cdot}$	2	-2	2	4

- **Note:** The vector of non-linear parameters correspond to the lower-diagonal elements of the choleski matrix of Σ (10).

How to account for endogenous characteristics?

- Two cases:
  - *Linear characteristics:* Replace x_{jt} with instrument w_{jt} when defining moment conditions (standard solution).
  - *Non-linear characteristics:* More difficult problem...
- Two approaches:
  1. Heuristic approximation to optimal IVs similar to BLP-1995
  2. Natural experiment-type variation (i.e. fixed-effects)

Example 1: Instruments for non-linear attributes

- Payoff function: Quality ladder

$$u_{ijt} = \delta_{jt} - \alpha_i p_{jt} + \varepsilon_{ijt}$$

where $\alpha_i = \sigma_p y_i^{-1}$, and $\log(y_i) \sim N(\mu_y, \sigma_y)$ (known).

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- BLP (1995): Prices and ξ_{jt} are simultaneously determined

$$E\left[\sigma_j^{-1}\left(\boldsymbol{s}_t, \boldsymbol{x}_t, \boldsymbol{p}_t | \sigma_p^0\right) | \boldsymbol{x}_t, \boldsymbol{w}_t\right] \neq E\left[\sigma_j^{-1}\left(\boldsymbol{s}_t, \boldsymbol{x}_t, \boldsymbol{p}_t | \sigma_p^0\right) | \boldsymbol{x}_t, \boldsymbol{p}_t\right]$$

where $\boldsymbol{w}_t = \{w_{jt}\}_{j=1,\dots,J_t}$ is a vector of excluded price instruments.

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where $\boldsymbol{w}_t = \{w_{jt}\}_{j=1,\dots,J_t}$ is a vector of excluded price instruments.

- **Curse of dimensionality:** Except in 'very' special cases (e.g. single-product Bertrand), the conditional distribution of prices is not a symmetric function of $(\boldsymbol{x}_t, \boldsymbol{w}_t)$.

$$E\left[\sigma_j^{-1}\left(\boldsymbol{s}_t, \boldsymbol{x}_t, \boldsymbol{p}_t | \sigma_p^0\right) | \boldsymbol{x}_t, \boldsymbol{w}_t\right] \neq g(\mathbf{d}_t^x, \mathbf{d}_t^w)$$

How to account for heterogenous price coefficient?

- **Heuristic solution:** Distribute the expectation for price *inside* of the inverse-demand function (Berry et al. 1999):

$$\begin{aligned} E\left[\sigma_j^{-1}(\boldsymbol{s}_t, \boldsymbol{p}_t, \boldsymbol{x}_t^{(2)}; \lambda) | \boldsymbol{x}_t, \boldsymbol{w}_t\right] &\approx E\left[\sigma_j^{-1}(\boldsymbol{s}_t, \hat{\boldsymbol{p}}_t, \boldsymbol{x}_t^{(2)}; \lambda) | \boldsymbol{x}_t, \hat{\boldsymbol{p}}_t\right] \\ &= g(\boldsymbol{d}_{jt}^{x}, \boldsymbol{d}_{jt}^{\hat{p}}) \end{aligned}$$

  where $\hat{d}_{jt,k}^{p} = E(p_{kt}|\boldsymbol{w}_{kt}) - E(p_{jt}|\boldsymbol{w}_{jt})$.

- $\hat{p}_{jt} = E(p_{jt}|\boldsymbol{w}_{kt})$ is the 'first-stage' predicted price.

Experiment 3: Differentiation IVs with Endogenous Prices

Example with cost shifter

1. Exogenous price index (OLS):

$$\hat{p}_{jt} = \hat{\pi}_0 + \hat{\pi}_1 x_{jt} + \hat{\pi}_2 \omega_{jt}$$

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2. Differentiation IV: Quadratic

$$\sum_{j'} \left(d^{\hat{p}}_{jt,j'} \right)^2 \text{ and } \sum_{j'} \left(d^{\hat{p}}_{jt,j'} \right)^2 \cdot \boldsymbol{d}_{jt,j'}$$

where $\boldsymbol{d}_{jt,j'} = (d^{x}_{jt,j'}, d^{\hat{p}}_{jt,j'})$.

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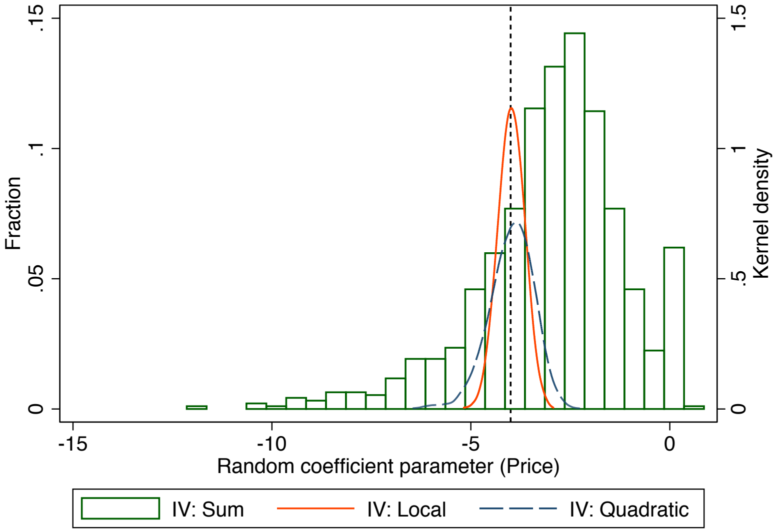
3. Differentiation IV: Local

$$\sum_{j'} \left(|d^{\hat{p}}_{jt,j'}| < \text{sd}(\hat{p}_{jt}) \right) \text{ and } \sum_{j'} \left(|d^{\hat{p}}_{jt,j'}| < \text{sd}(\hat{p}_{jt}) \right) \cdot \boldsymbol{d}_{jt,j'}$$

Distribution of $\hat{\sigma}_p$ with weak and strong IVs



Dash vertical line = True parameter value

GMM estimates with endogenous prices

	(1) True	(2) Diff. IV = Local			(3) Diff. IV = Quadratic			(4) Diff. IV = Sum		
		bias	se	rmse	bias	se	rmse	bias	se	rmse
λ_p	-4.00	0.02	0.27	0.28	0.02	0.53	0.55	1.03	158.25	2.10
β_p	-0.20	0.01	0.37	0.37	0.01	0.31	0.32	-0.67	201.29	1.38
β_0	50.00	-0.26	3.92	3.92	-0.28	7.36	7.45	-9.82	26.41	20.65
β_x	2.00	-0.02	0.46	0.45	-0.02	0.47	0.47	0.34	1.11	0.83

GMM estimates with endogenous prices

	(1) IV = Local	(2) IV=Quadratic	(3) IV = Sum
Frequency conv.	1	1	0.94
IIA-test	109.48	53.90	1.88
p-value	0	0	0.34
1st-stage F-test: Price	191.80	442.10	138.94
1st-stage F-test: Jacobian	214.60	58.40	27.85
Cond. 1st-stage F-test: Price	252.23	479.96	7.92
Cond. 1st-stage F-test: Jacobian	280.31	82.44	6.19
Cragg-Donald statistics	170.19	54.45	4.09
Stock-Yogo size CV (10%)	16.87	13.43	13.43
Nb. endogenous variables	2	2	2
Nb. IVs	4	3	3

- The *Conditional 1st-stage F-test* statistic is the Weak IV test proposed by Angrist and Pischke for multiple endogenous variables.
- The IIA test is testing the exclusion restriction, $H_0 : \gamma = 0$ from the following linear IV regression:

$$\ln s_{jt}/s_{0t} = x_{jt}\beta + \alpha p_{jt} + \gamma IV_{jt}^{diff} + u_{jt}$$

where (β, α, γ) are estimated by GMM using the cost-shifter (ω_{jt}) as excluded instrument.

Example 2: Natural Experiments

- An alternative solution is to exploit natural experiments that vary the choice-set over time or across markets.
- Example: Three-way panel, product j, market m, and time $(t = 0, 1)$.

$$\text{Simultaneity problem: } E[\xi_{jmt} | \mathbf{x_{mt}}] \neq \mathbf{0}$$

$$\text{Decomposition: } \xi_{jmt} = \mu_{jm} + \tau_t + \Delta\xi_{mt}$$

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- **Assumption:** Quasi-experimental design

$$E[\Delta\xi_{jmt} | \bar{\xi}_m, \tau_t, \mathbf{x_{mt}}] = \mathbf{0}$$

Experiment 4: Random Entry in Hotelling

- **Hotelling example:** Exogenous entry of a new product ($x' = 5$)

$$u_{ijmt} = \delta_{jmt} - \lambda|\nu_i - x_{jmt}| + \epsilon_{ijmt}$$

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- Reduced-form: Difference-in-difference regression

$$\sigma_j^{-1}(s_t, x_t|\theta^0) = \mu_{jm} + \tau_t + \gamma D_{jm} \times 1(t = 1) + \xi_{jmt}$$

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- **GMM:** DiD IVs
  - Linear characteristics: $x_{jmt}^{(1)}$ = Market/Product FE + After Dummy
  - Differentiation IV: $z_{jmt} = D_{jm} \times 1(t = 1)$
  - $\hat{\theta}^{gmm}$ is identified from the DiD variation in z_{jmt}.

Natural Experiment: Hotelling Example

DGP: $\delta_{jmt} = \bar{\xi}_{jm} + \tau_t + \Delta\xi_{jmt}$, where $E(\bar{\xi}_{jm}|\boldsymbol{x}_m) \neq 0$

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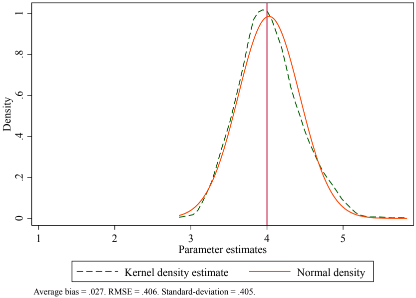
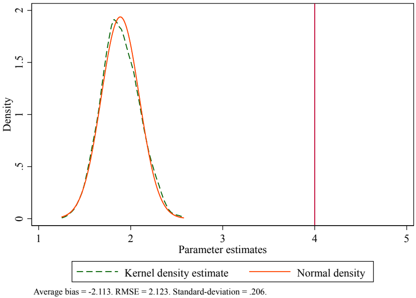
Difference-in-Difference Moments



Differentiation IVs w/o FEs



- "Diff-in-Diff" specification:

$$\boldsymbol{z}_{jmt} = \{\text{Product/Market FE}_{jm}, 1(t=1), 1(|x_{jm} - 5| < 1)1(t=1)\}$$

Optimal IV Approximation

- Abstracting from heteroscedasticity concerns, the "Optimal IV" takes the following form:

$$A_j^*(\boldsymbol{x}_t) = E\left[\frac{\partial \rho_j(\boldsymbol{s}_t, \boldsymbol{x}_t; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \Big| \boldsymbol{x}_t\right] = \left\{-\boldsymbol{x}_{jt}, E\left[\frac{\partial \sigma_j^{-1}(\boldsymbol{s}_t, \boldsymbol{x}_t^{(2)}; \boldsymbol{\theta})}{\partial \boldsymbol{\lambda}} \Big| \boldsymbol{x}_t\right]\right\}$$

- Instead of using non-parametric regressions to approximate $A_j^*(\boldsymbol{x}_t)$, Berry et al. (1999) propose the following heuristic:

$$\tilde{A}_j(\boldsymbol{x}_t|\boldsymbol{\theta}) = \frac{\partial \sigma_j^{-1}(\boldsymbol{s}_t, \boldsymbol{p}_t, \boldsymbol{x}_t^{(2)}; \boldsymbol{\theta})}{\partial \boldsymbol{\lambda}}\Bigg|_{p_{jt}=\hat{p}_{jt}, \xi_{jt}=0, \forall j,t}$$

  where $\hat{p}_{jt} \approx E(p_{jt}|\boldsymbol{x}_t, \boldsymbol{w}_t)$ is a "reduced-form" model for prices independent of ξ_{jt}.

- This leads to a two-step estimator:
  - Obtain initial estimate $\hat{\theta}^1$ using instrument vector $A_j(\boldsymbol{x}_t)$
  - Compute $\tilde{A}_j\left(\boldsymbol{x}_t|\hat{\boldsymbol{\theta}}^1\right)$, and re-estimate the model (just-identified).

Optimal IV Approximation

- Reynaert and Verboven (2013) show that this procedure improves substantially the weak IV problems.
- **Alternative approach:** Exploit the property that the optimal IV is a symmetric function of the vector of characteristics differences.
  - Use **Differentiation IVs** to obtain θ^1
  - Approximate the optimal IV directly by projecting the Jacobian on $A_j(\boldsymbol{x}_t)$ (Newey 1990)
- **Questions:**
  - How important is it to use consistent first-stage estimates to construct a valid Optimal IV approximation?
  - What is the efficiency gain of using optimal IV heuristic, relative to using differentiation IVs directly?

Optimal IV approximation with alternative initial parameter values

	Normal RC			Hotelling		
	λ^1	bias	rmse	λ^1	bias	rmse
Optimal IV approx.:						
(1)	0.5	0.001	0.027	4	-0.003	0.140
(2)	1.5	0.001	0.026	**2**	**-0.004**	**0.126**
(3)	2	**0.001**	**0.026**	0	-0.079	0.509
(3)	2.5	0.001	0.026	-1	-0.344	1.687
(4)	3	0.002	0.028	-2	-0.282	1.254
Differentiation IV	—	0.001	0.031	—	0.017	0.310

- **Takeaway 1:** With IID RC, inconsistent first-stage does not lead to biased or noisy estimates. The optimal IV approximation is "strong" for all λ^1!
- **Takeaway 2:** With the hotelling model, inconsistent first-stage leads to biased estimates and weak instruments.
- Why? The magnitude of λ does not determine "who competes with who". Only the magnitude of diversions.

Example 2: Correlated Random Coefficients

Choleski matrix	True (1)	Opt. IV: $\boldsymbol{\theta}^1 \sim N(0,1)$ bias (2)	rmse (3)	se (4)	Opt. IV: $\boldsymbol{\theta}^1 \sim N(0,4)$ bias (4)	rmse (5)	se (6)	Diff. IV: Quad. bias (7)	rmse (8)	se (9)
$\log c_{11}$	0.69	0.00	0.22	5.42	0.01	1.22	11.92	-0.00	0.03	0.03
$\log c_{22}$	0.55	-0.01	0.19	2.50	-0.16	2.36	192.70	-0.00	0.04	0.04
$\log c_{33}$	0.49	-0.02	0.15	0.46	-0.44	2.69	++	-0.00	0.04	0.04
$\log c_{44}$	0.46	-0.22	1.83	++	-1.78	5.57	++	-0.00	0.04	0.04
c_{21}	-1.00	0.01	0.47	4.51	0.03	0.77	781.85	0.00	0.06	0.06
c_{31}	1.00	0.00	0.33	0.86	-0.02	0.63	23.48	-0.00	0.07	0.07
c_{32}	-0.58	0.02	0.27	2.69	0.03	0.56	285.80	0.00	0.07	0.08
c_{41}	1.00	0.00	0.23	1.37	0.00	0.58	333.93	0.00	0.07	0.07
c_{42}	-0.58	0.01	0.23	2.69	0.04	0.50	484.88	0.00	0.08	0.08
c_{43}	0.41	0.00	0.23	1.59	0.03	0.52	++	0.00	0.08	0.08

Example 3: Efficiency gains

Quality ladder model

		True	Diff. IV = Local bias	Diff. IV = Local se	Diff. IV = Local rmse	Diff. IV = Quadratic bias	Diff. IV = Quadratic se	Diff. IV = Quadratic rmse	Diff. IV = Sum bias	Diff. IV = Sum se	Diff. IV = Sum rmse
1st-stage	λ_p	-4	0.02	0.27	0.28	0.02	0.53	0.55	1.01	2.66	2.09
	β_0	50	-0.26	3.92	3.92	-0.28	7.36	7.45	-9.63	26.48	20.46
	β_x	2	-0.02	0.46	0.45	-0.02	0.47	0.47	0.34	1.11	0.83
	β_p	-0.2	0.01	0.37	0.37	0.01	0.31	0.32	-0.66	1.76	1.37
2nd-stage	λ_p	-4	0.00	0.24	0.23	0.00	0.24	0.23	0.01	0.26	0.31
	β_0	50	-0.07	3.99	3.84	-0.06	3.72	3.65	0.05	4.32	4.61
	β_x	2	-0.01	0.48	0.47	-0.01	0.41	0.41	0.03	0.52	0.51
	β_p	-0.2	0.01	0.36	0.36	0.00	0.31	0.32	-0.03	0.40	0.40

Conclusion

- What did we do:
  - Show how that the characteristic model can be used to construct **relevant** instruments to identify substitution patterns
  - And, eliminate the weak IV problem that is present in applied work
  - *Differentiation IV's*: Capture the relative position of each product in the characteristic space.

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  - Weak IV tests

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- What's next?
  - Higher-order basis: Lasso
  - Conduct tests
  - Non-parametric estimation

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